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SOLUTIONS OF THE EXAMPLES

IN

HALL AND KNIGHT'S

ELEMENTARY TRIGONOMETRY.

SOLUTIONS OF THE EXAMPLES

IN

HALL AND KNIGHT'S

ELEMENTARY TRIGONOMETRY

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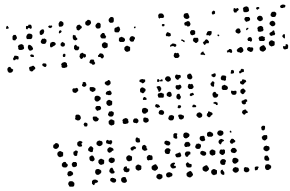


PREFACE.

In preparing this Key two objects have been kept in view. It is intended first to save the time and lighten the work of teachers, and secondly to afford help to those who study Mathematics without the guidance of a teacher. Accordingly the solutions have generally been given in the most simple and natural manner, with frequent reference to the text and examples in the *Elementary Trigonometry*. In particular, the solutions which involve logarithmic work have been presented in the fullest detail, so that with the help of the Key, a teacher will be able very readily to discover and correct mistakes in the work of his pupils.

For very many of the solutions I am indebted to Mr H. C. Playne of Clifton College, and my thanks are due to him for valuable help all through the book.

H. S. HALL.

January, 1895.

ELEMENTARY TRIGONOMETRY.

EXAMPLES. I. PAGE 4.

7. $69° 13' 30'' = \cdot 76916$ of a right angle $= 76^g 91' 66{\cdot}7''$.

8. $19° 0' 45'' = \cdot 21125$ of a right angle $= 21^g 12' 50''$.

9. $50° 37' 5{\cdot}7'' = \cdot 562425$ of a right angle $= 56^g 24' 25''$.

10. $43° 52' 38{\cdot}1'' = \cdot 487525$ of a right angle $= 48^g 75' 25''$.

11. $11° 0' 38{\cdot}4'' = \cdot 1223407$ of a right angle $= 12^g 23' 40{\cdot}7''$.

12. $142° 15' 45'' = 1{\cdot}5806944$ of a right angle $= 158^g 6' 94{\cdot}4''$.

13. $12' 9'' = \cdot 00225$ of a right angle $= 22' 50''$.

14. $3' 26{\cdot}3'' = \cdot 000636$ of a right angle $= 6' 36{\cdot}7''$.

15. $56^g 87' 50'' = \cdot 56875$ of a right angle $= 51° 11' 15''$.

16. $39^g 6' 25'' = \cdot 390625$ of a right angle $= 35° 9' 22{\cdot}5''$.

17. $40^g 1' 25{\cdot}4'' = \cdot 4001254$ of a right angle $= 36° 0' 40{\cdot}6''$.

18. $1^g 2' 3'' = \cdot 010203$ of a right angle $= 55' 5{\cdot}8''$.

19. $3^g 2' 55'' = \cdot 030205$ of a right angle $= 2° 43' 6{\cdot}4''$.

20. $8^g 10' 6{\cdot}5'' = \cdot 0810065$ of a right angle $= 7° 17' 26{\cdot}1''$.

21. $6' 25'' = \cdot 000625$ of a right angle $= 3' 22{\cdot}5''$.

22. $37' 5'' = \cdot 003705$ of a right angle $= 20' 0{\cdot}4''$.

23. Let the angles expressed in degrees be A and B;

then $$A + B = \frac{9}{10} \times 80° = 72°, \text{ and } A - B = 18°.$$

Hence $A = 45°$, $B = 27°$.

24. If n is the number of degrees in the angle, $n + \frac{10}{9}n = 152$, whence $n =$

25. Here $\frac{x}{60}$ = number of degrees, and $\frac{y}{100}$ = number of grades.
Therefore $\frac{x}{60} = \frac{9}{10} \cdot \frac{y}{100}$; whence we obtain $50x = 27y$.

26. Here $\frac{s}{60 \times 60}$ = number of degrees, and $\frac{t}{100 \times 100}$ = numbe
the angle. Therefore $\frac{s}{36} = \frac{9}{10} \times \frac{t}{100}$; that is, $250s = 81t$.

EXAMPLES. II. PAGE 11.

[The following five solutions will sufficiently illustrate this e

3. From fig. of Art. 17 we have $a^2 = b^2 + c^2 = 400 + 225 =$
$a = 25$, and $\sin C = \frac{4}{5}$, $\cos B = \frac{4}{5}$, $\cot C = \frac{3}{4}$, $\sec C = \frac{5}{3}$.

6. Let $a = 15$, $b = 9$; then $c^2 = a^2 - b^2 = (a+b)(a-b) = 24$
$c = 12$, and $\sin C = \frac{4}{5}$, $\cos C = \frac{3}{5}$, $\tan C = \frac{4}{3}$.

8. In the third fig. of Art. 17, let $AC = 41$, $AB = 9$;
then $$BC^2 = 41^2 - 9^2 = (41+9)(41-9);$$
whence $BC = 40$, and $\sin A = \frac{40}{41}$, $\cot A = \frac{9}{40}$.

10. Here $CD = 2DE$. Hence if $ED = a$, $DC = 2a$, EC
required ratios may now be written down.

11. From the right-angled $\triangle ABC$ we have $BC = 39$; also fr
angled $\triangle ACD$ we have $DC = 77$. The required ratios may nc
down.

EXAMPLES. IIIa. PAGE 17.

Examples 1—25 are too easy to require full solution; the fc
solutions will suffice.

10. $(1 - \cos^2\theta)\sec^2\theta = \sin^2\theta \times \frac{1}{\cos^2\theta} = \tan^2\theta$.

12. $\operatorname{cosec}\alpha \sqrt{1 - \sin^2\alpha} = \frac{1}{\sin\alpha} \times \cos\alpha = \cot\alpha$.

15. $(1 - \cos^2\theta)(1 + \tan^2\theta) = \sin^2\theta \sec^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$.

$(1 - \cos^2 A)(1 + \cot^2 A) = \sin^2 A \operatorname{cosec}^2 A = 1$.

). $\sin\alpha\sec\alpha\sqrt{\operatorname{cosec}^2\alpha-1}=\dfrac{\sin\alpha}{\cos\alpha}\times\cot\alpha=1.$

). $\sin^2\theta\cot^2\theta+\sin^2\theta=\sin^2\theta(1+\cot^2\theta)=\sin^2\theta\operatorname{cosec}^2\theta=1.$

). $(1+\tan^2\theta)(1-\sin^2\theta)=\sec^2\theta\cos^2\theta=1.$

). $\operatorname{cosec}^2\theta\tan^2\theta-1=\dfrac{1}{\sin^2\theta}\cdot\dfrac{\sin^2\theta}{\cos^2\theta}-1=\sec^2\theta-1=\tan^2\theta.$

). First side $=\cos^2A+\sin^2A=1.$

'. First side $=\sec^2A-\tan^2A=1.$

). First side $=\sin A\,.\sin A+\cos A\,.\cos A=\sin^2A+\cos^2A=1.$

). First side $=\sec A\,.\sec A-\tan A\,.\tan A=\sec^2A-\tan^2A=1.$

). $\sin^4\alpha-\cos^4\alpha=(\sin^2\alpha+\cos^2\alpha)(\sin^2\alpha-\cos^2\alpha)$
$=\sin^2\alpha-\cos^2\alpha=\sin^2\alpha-(1-\sin^2\alpha)$
$=2\sin^2\alpha-1.$

so $\sin^2\alpha-\cos^2\alpha=1-\cos^2\alpha-\cos^2\alpha=1-2\cos^2\alpha.$

L. First side $=(\sec^2\alpha-1)(\sec^2\alpha+1)=\tan^2\alpha\,(2+\tan^2\alpha).$

). First side $=(\operatorname{cosec}^2\alpha-1)(\operatorname{cosec}^2\alpha+1)=\cot^2\alpha\,(\cot^2\alpha+2).$

). First side $=\left(\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{1}{\sin\alpha}\right)^2-\left(\dfrac{\sin\alpha}{\cos\alpha}\right)^2=\sec^2\alpha-\tan^2\alpha.$

L. First side $=\left(\dfrac{1}{\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta}\right)^2-\left(\dfrac{\cos\theta}{\sin\theta}\right)^2=\operatorname{cosec}^2\theta-\cot^2\theta.$

). $\operatorname{cosec}^2\theta-\cot^2\theta=1=\sec^2\theta-\tan^2\theta.$ Transpose.

EXAMPLES. III b. PAGE 19.

L. $\operatorname{vers}\theta\sec\theta=(1-\cos\theta)\sec\theta=\sec\theta-1.$

). First side $=\dfrac{1}{\cos\theta}-\dfrac{\sin\theta}{\cos\theta}\sin\theta=\dfrac{1-\sin^2\theta}{\cos\theta}=\dfrac{\cos^2\theta}{\cos\theta}=\cos\theta.$

). First side $=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}=\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$
$=\dfrac{1}{\sin\theta\cos\theta}=\operatorname{cosec}\theta\sec\theta.$

'. First side $=\operatorname{cosec}A\tan A\cos A=\dfrac{\cos A}{\sin A}\tan A=1.$

). First side $=\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta+\cos^2\theta+\sin^2\theta-2\cos\theta\sin\theta$
$=1+1=2.$

). First side $=1+2\tan\theta+\tan^2\theta+1-2\tan\theta+\tan^2\theta$
$=2+2\tan^2\theta=2(1+\tan^2\theta)=2\sec^2\theta.$

10. First side $= \cot^2\theta - 2\cot\theta + 1 + \cot^2\theta + 2\cot\theta + 1$
$= 2\cot^2\theta + 2 = 2(\cot^2\theta + 1) = 2\operatorname{cosec}^2\theta.$

11. First side $= \sin^2 A \operatorname{cosec}^2 A + \cos^2 A \sec^2 A = 1 + 1 = 2.$

12. First side $= \cos^2 A \times 1 + \sin^2 A \times 1 = 1.$

13. First side $= \cot^2\alpha(1 + \cot^2\alpha) = (\operatorname{cosec}^2\alpha - 1)\operatorname{cosec}^2\alpha$
$= \operatorname{cosec}^4\alpha - \operatorname{cosec}^2\alpha.$

14. First side $= \dfrac{\tan^2\alpha}{\sec^2\alpha} \cdot \dfrac{\operatorname{cosec}^2\alpha}{\cot^2\alpha} = \dfrac{\sin^2\alpha}{\cos^2\alpha\sec^2\alpha} \times \dfrac{\operatorname{cosec}^2\alpha \sin^2\alpha}{\cos^2\alpha}$
$= \dfrac{\sin^2\alpha}{\cos^2\alpha} = \sin^2\alpha\sec^2\alpha.$

15. First side $= \dfrac{1 + \sin\alpha + 1 - \sin\alpha}{1 - \sin^2\alpha} = \dfrac{2}{\cos^2\alpha} = 2\sec^2\alpha.$

16. First side $= \dfrac{\tan\alpha(\sec\alpha + 1) + \tan\alpha(\sec\alpha - 1)}{\sec^2\alpha - 1} = \dfrac{2\tan\alpha\sec\alpha}{\tan^2\alpha}$
$= \dfrac{2\sec\alpha}{\tan\alpha} = \dfrac{2\sec\alpha\cos\alpha}{\sin\alpha} = \dfrac{2}{\sin\alpha} = 2\operatorname{cosec}\alpha.$

17. First side $= \dfrac{1}{1 + \sin^2\alpha} + \dfrac{1}{1 + \dfrac{1}{\sin^2\alpha}} = \dfrac{1}{1 + \sin^2\alpha} + \dfrac{\sin^2\alpha}{1 + \sin^2\alpha}$
$= \dfrac{1 + \sin^2\alpha}{1 + \sin^2\alpha} = 1.$

18. First side $= \left(\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}\right)(\sin\theta + \cos\theta)$
$= \dfrac{(\sin\theta + \cos\theta)(\sin\theta + \cos\theta)}{\cos\theta\sin\theta}$
$= \dfrac{\sin^2\theta + \cos^2\theta + 2\cos\theta\sin\theta}{\cos\theta\sin\theta} = \dfrac{1 + 2\cos\theta\sin\theta}{\cos\theta\sin\theta}$
$= \dfrac{1}{\cos\theta\sin\theta} + 2 = \sec\theta\cos\theta + 2.$

20. First side $= (1 + \cot\theta)^2 - \operatorname{cosec}^2\theta = 1 + 2\cot\theta + \cot^2\theta - \operatorname{cosec}^2\theta$
$= 1 + 2\cot\theta - 1 = 2\cot\theta.$

22. First side
$= \sin^2 A + 2\sin A \operatorname{cosec} A + \operatorname{cosec}^2 A + \cos^2 A + 2\cos A \sec A + \sec^2 A$
$= (\sin^2 A + \cos^2 A) + 2 + (\cot^2 A + 1) + 2 + (\tan^2 A + 1)$
$= \tan^2 A + \cot^2 A + 7.$

23. First side $= (2\sec^2 A - 1)(2\operatorname{cosec}^2 A - 1)$
$= 4\sec^2 A \operatorname{cosec}^2 A - 2\sec^2 A - 2\operatorname{cosec}^2 A + 1$
$= 1 + 4\sec^2 A \operatorname{cosec}^2 A - 2(\sec^2 A \operatorname{cosec}^2 A)$ [Art. 31, Ex. 1.]
$= 1 + 2\sec^2 A \operatorname{cosec}^2 A.$

24. First side $= 1 + (\sin^2 A + \cos^2 A) - 2\sin A + 2\cos A - 2\sin A\cos A$
$= 2[1 - \sin A + \cos A - \sin A\cos A]$
$= 2(1 - \sin A)(1 + \cos A).$

25. First side $= \sin A\left(1 + \dfrac{\sin A}{\cos A}\right) + \cos A\left(1 + \dfrac{\cos A}{\sin A}\right)$
$= \sin A\,\dfrac{(\cos A + \sin A)}{\cos A} + \cos A\,\dfrac{(\sin A + \cos A)}{\sin A}$
$= \tan A\,(\sin A + \cos A) + \cot A\,(\sin A + \cos A)$
$= (\sin A + \cos A)\left(\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}\right)$
$= \dfrac{(\sin A + \cos A)(\sin^2 A + \cos^2 A)}{\sin A\cos A}$
$= \dfrac{\sin A + \cos A}{\sin A\cos A} = \dfrac{1}{\cos A} + \dfrac{1}{\sin A} = \sec A + \operatorname{cosec} A.$

26. First side $= \cos\theta\,(2\tan^2\theta + 5\tan\theta + 2) = \dfrac{2\sin^2\theta}{\cos^2\theta}\cos\theta + 5\sin\theta + 2\cos\theta$
$= 2\left(\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta\right) + 5\sin\theta = \dfrac{2(\sin^2\theta + \cos^2\theta)}{\cos\theta} + 5\sin\theta$
$= 2\sec\theta + 5\sin\theta.$

27. First side $= \left(\dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta}\right)^2 = \left(\dfrac{1 + \sin\theta}{\cos\theta}\right)^2$
$= \dfrac{(1 + \sin\theta)(1 + \sin\theta)}{1 - \sin^2\theta} = \dfrac{(1 + \sin\theta)(1 + \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)} = \dfrac{1 + \sin\theta}{1 - \sin\theta}.$

28. First side $= \dfrac{\cos\theta\,(2\sin\theta - 1)}{2\sin^2\theta - \sin\theta} = \dfrac{\cos\theta\,(2\sin\theta - 1)}{\sin\theta\,(2\sin\theta - 1)} = \cot\theta.$

29. First side $= \dfrac{\cot^2\theta\,(\sec\theta - 1)}{1 + \sin\theta} - \dfrac{\sec^2\theta\,(1 - \sin\theta)}{1 + \sec\theta}$
$= \dfrac{\cot^2\theta\,(\sec^2\theta - 1) - \sec^2\theta\,(1 - \sin^2\theta)}{(1 + \sin\theta)(1 + \sec\theta)}$
$= \dfrac{\cot^2\theta\tan^2\theta - \sec^2\theta\cos^2\theta}{(1 + \sin\theta)(1 + \sec\theta)} = 0.$

30. $\tan^2\alpha + \sec^2\beta = (\sec^2\alpha - 1) + (\tan^2\beta + 1) = \sec^2\alpha + \tan^2\beta.$

31. $\dfrac{\tan\alpha + \cot\beta}{\cot\alpha + \tan\beta} = \dfrac{\tan\alpha + \dfrac{1}{\tan\beta}}{\dfrac{1}{\tan\alpha} + \tan\beta} = \dfrac{\tan\alpha\tan\beta + 1}{\tan\beta} \cdot \dfrac{\tan\alpha}{\tan\alpha\tan\beta + 1} = \dfrac{\tan\alpha}{\tan\beta}.$

33. First side $= \cot\alpha\tan\alpha\tan\beta + \tan\beta\cot\beta\cot\alpha$
$= \tan\beta + \cot\alpha.$

34. First side $= \sin^2\alpha\,(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta$
$= \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta$
$= \sin^2\alpha - \sin^2\beta.$

35. First side $= (1+\tan^2\alpha)\tan^2\beta - \tan^2\alpha\,(1+\tan^2\beta)$
$= \tan^2\beta + \tan^2\alpha\tan^2\beta - \tan^2\alpha - \tan^2\alpha\tan^2\beta$
$= \tan^2\beta - \tan^2\alpha.$

36. First side $= \sin^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta$, the other terms cancelling;
this expression $= (\sin^2\alpha+\cos^2\alpha)\cos^2\beta + (\cos^2\alpha+\sin^2\alpha)\sin^2\beta$
$= \cos^2\beta + \sin^2\beta = 1.$

EXAMPLES. III. c. PAGE 23.

1. $\sec A = \dfrac{1}{\cos A} = \dfrac{1}{\sqrt{1-\sin^2 A}} = \dfrac{1}{\sqrt{1-\frac{1}{4}}} = \dfrac{2}{\sqrt{3}}.$

$\cot A = \dfrac{\cos A}{\sin A} = \dfrac{\sqrt{1-\sin^2 A}}{\sin A} = \dfrac{\sqrt{3}}{2} \div \dfrac{1}{2} = \sqrt{3}.$

2. $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}$ [Art. 32, Ex. 1]

$= \dfrac{4}{3} \div \sqrt{1+\dfrac{16}{9}} = \dfrac{4}{3} \times \dfrac{3}{5} = \dfrac{4}{5}.$

$\cos A = \cot A \,.\, \sin A = \dfrac{3}{4} \cdot \dfrac{4}{5} = \dfrac{3}{5}.$

5. $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\dfrac{1}{49}} = \sqrt{\dfrac{48}{49}} = \dfrac{\sqrt{48}}{7}.$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{7} \div \dfrac{\sqrt{48}}{7} = \dfrac{1}{\sqrt{48}}.$

6. $\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\dfrac{7^2}{25^2}} = \dfrac{24}{25}.$ Therefore $\sec A = \dfrac{25}{24}.$

$\tan A = \dfrac{\sin A}{\cos A} = \dfrac{7}{25} \times \dfrac{25}{24} = \dfrac{7}{24}.$

8. $\operatorname{cosec}\alpha = \sqrt{1+\cot^2\alpha}.$ [Art. 27.]

$\cos\alpha = \cot\alpha \sin\alpha = \dfrac{\cot\alpha}{\operatorname{cosec}\alpha} = \dfrac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}.$

10. $\operatorname{cosec} A = \dfrac{1}{\sin A}$; $\cos A = \sqrt{1-\sin^2 A}$; $\sec A = \dfrac{1}{\sqrt{1-\sin^2 A}}$.

$\tan A = \dfrac{\sin A}{\cos A} = \dfrac{\sin A}{\sqrt{1-\sin^2 A}}$; $\cot A = \dfrac{\sqrt{1-\sin^2 A}}{\sin A}$.

11. Here $\sin A = \cos A$, so that $\tan A = 1$.

$\therefore$ $\operatorname{cosec} A = \sqrt{1+\cot^2 A} = \sqrt{1+1} = \sqrt{2}$.

12. $\tan A = \dfrac{\sin A}{\sqrt{1-\sin^2 A}} = \dfrac{m}{n} \div \sqrt{1-\dfrac{m^2}{n^2}}$

$= \dfrac{m}{n} \times \dfrac{n}{\sqrt{n^2-m^2}} = \dfrac{m}{\sqrt{n^2-m^2}}$.

13. $p^2 \cot^2\theta = q^2 - p^2$; $\therefore$ $p^2(\cot^2\theta + 1) = q^2$.

$\therefore$ $p^2 \operatorname{cosec}^2\theta = q^2$, so that $\sin\theta = \dfrac{p}{q}$.

14. In the diagram of Ex. 2, Art. 33, let $PQ = 2m$, $PR = m^2+1$; then

$RQ^2 = (m^2+1)^2 - (2m)^2 = (m^2-1)^2$.

$\therefore$ $RQ = m^2 - 1$.

$\therefore$ $\tan A = \dfrac{m^2-1}{2m}$, $\sin A = \dfrac{m^2-1}{m^2+1}$.

16. The expression $= \dfrac{2\tan\alpha - 3}{4\tan\alpha - 9}$; but $\tan\alpha = \sqrt{\dfrac{169}{25} - 1} = \dfrac{12}{5}$.

$\therefore$ the expression $= \dfrac{\dfrac{2 \cdot 12}{5} - 3}{\dfrac{4 \cdot 12}{5} - 9} = \dfrac{9}{3} = 3$.

17. The expression $= \dfrac{p\cot\theta - q}{p\cot\theta + q} = \dfrac{\dfrac{p^2}{q} - q}{\dfrac{p^2}{q} + q} = \dfrac{p^2 - q^2}{p^2 + q^2}$.

EXAMPLES. IV. a. PAGE 26.

Let E stand for the expression to be evaluated in each case; then

6. $E = (1)^2 \times \dfrac{\sqrt{3}}{2} \cdot \dfrac{1}{\sqrt{3}} (\sqrt{3})^2 = \dfrac{3}{2}$.

7. $E = (\sqrt{3})^2 + 4\left(\dfrac{1}{\sqrt{2}}\right)^2 + 3\left(\dfrac{2}{\sqrt{3}}\right)^2 = 3 + \dfrac{4}{2} + 4 = 9$.

8. $E=\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2+(\sqrt{2})^2-2\left(\frac{1}{\sqrt{3}}\right)^2=\frac{1}{2}\cdot\frac{4}{3}+2-\frac{2}{3}=2.$

9. $E=\left(\frac{1}{\sqrt{3}}\right)^2+2\left(\frac{\sqrt{3}}{2}\right)+1-\sqrt{3}+\left(\frac{\sqrt{3}}{2}\right)^2=\frac{1}{3}+\sqrt{3}+1-\sqrt{3}+\frac{3}{4}=\frac{25}{12}.$

10. $E=(1)^2+\frac{1}{2}-\left(\frac{\sqrt{3}}{2}\right)^2-\frac{3}{4}\left(\frac{1}{\sqrt{3}}\right)^2=1+\frac{1}{2}-\frac{3}{4}-\frac{1}{4}=\frac{1}{2}.$

11. $E=3\left(\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{2}(\sqrt{2})^2-\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^2=1+1-1-\frac{1}{4}=\frac{3}{4}.$

12. $E=\frac{1}{2}-1^2+\frac{3}{4}\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{2}=\frac{1}{2}-1+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}=-\frac{1}{2}.$

13. $E=\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{2}\cdot 2\left(\frac{1}{\sqrt{3}}\right)^2+\frac{4}{3}\left(\frac{1}{\sqrt{2}}\right)^2(\sqrt{3})^2=\frac{1}{4}-\frac{1}{3}+2=1\frac{11}{12}.$

14. We have

$$(1)^2-\left(\frac{1}{2}\right)^2=x\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{3};$$

$$1-\frac{1}{4}=x\cdot\frac{\sqrt{3}}{2};$$

$$\therefore x=\frac{\sqrt{3}}{2}.$$

15. We have

$$x\cdot\frac{1}{2}\cdot\left(\frac{1}{\sqrt{2}}\right)^2=\frac{(\sqrt{3})^2\cdot 2\cdot 1}{(\sqrt{2})^2\,2};$$

$$\frac{x}{4}=\frac{3}{2};$$

$$\therefore\quad x=6.$$

EXAMPLES. IV. b. PAGE 28.

For Examples 9—14, see Example 1, page 28.

20. Second side $=1+\sin^2 A\sec^2 A=1+\tan^2 A=\sec^2 A=\operatorname{cosec}^2(90^\circ-A)$.

21. First side $=\sin A\cot A\tan A\operatorname{cosec} A=1$.

22. First side $=\operatorname{cosec} A-\cot A\sin A\cot A=\operatorname{cosec} A\,(1-\cos^2 A)=\sin A$.

23. First side $=\tan^2 A\operatorname{cosec}^2 A-\sin^2 A\sec^2 A$

$=\sec^2 A-\tan^2 A=1$.

24. First side

$$=\cot A+\tan A=\frac{\cos^2 A+\sin^2 A}{\sin A\cos A}=\operatorname{cosec} A\sec A=\operatorname{cosec} A\operatorname{cosec}(90^\circ-A).$$

25. First side $=\dfrac{\cos A}{\operatorname{cosec} A}\cdot\dfrac{\cot A}{\cos A}=\cos A$.

26. First side $=\dfrac{\operatorname{cosec}^2 A\tan^2 A}{\tan A}\cdot\dfrac{\cot A}{\sec^2 A}=\cot^2 A$.

$=\operatorname{cosec}^2 A-1=\sec^2(90^\circ-A)-1$.

27. First side $= \dfrac{\tan A}{\operatorname{cosec}^2 A} \cdot \dfrac{\sec A \cot^3 A}{\cos^2 A} = \sec A = \sqrt{\tan^2 A + 1}$.

28. First side $= \dfrac{\sin^2 A}{1-\cos A} = \dfrac{1-\cos^2 A}{1-\cos A} = 1 + \cos A = 1 + \sin(90° - A)$.

29. First side $= \dfrac{\cot^2 A \cos^3 A}{\cot A (1+\sin A)} = \cot A (1 - \sin A) = \tan(90° - A) - \cos A$.

30. $x \cos A \tan A = \sin A$;
$\therefore \quad x = 1$.

31. $\sec^2 A - x \tan A = 1$;
$\therefore \quad \tan^2 A = x \tan A$;
$\therefore \quad x = \tan A$.

EXAMPLES. IV. c. PAGE 31.

9. $1 + \tan^2\theta = 2\tan^2\theta$;
$\tan^2\theta = 1$;
$\therefore \quad \tan\theta = \pm 1$;
$\therefore \quad \theta = 45°$.

10. $1 + \cot^2\theta = 4\cot^2\theta$;
$3\cot^2\theta = 1$;
$\therefore \quad \cot\theta = \pm\dfrac{1}{\sqrt{3}}$;
$\therefore \quad \theta = 60°$.

11. $1 + \tan^2\theta = 3\tan^2\theta - 1$;
$2\tan^2\theta = 2$;
$\therefore \quad \tan\theta = \pm 1$;
$\therefore \quad \theta = 45°$.

12. $1 + \tan^2\theta + \tan^2\theta = 7$;
$2\tan^2\theta = 6$;
$\therefore \quad \tan\theta = \pm\sqrt{3}$;
$\therefore \quad \theta = 60°$.

13. $\cot^2\theta + 1 + \cot^2\theta = 3$;
$\cot^2\theta = 1$;
$\therefore \quad \cot\theta = \pm 1$;
$\therefore \quad \theta = 45°$.

14. $2(\cos^2\theta - 1 + \cos^2\theta) = 1$;
$4\cos^2\theta = 3$;
$\therefore \quad \cos\theta = \pm\dfrac{\sqrt{3}}{2}$;
$\therefore \quad \theta = 30°$.

15. $2\cos^2\theta + 4 - 4\cos^2\theta = 3$;
$2\cos^2\theta = 1$;
$\therefore \quad \cos\theta = \pm\dfrac{1}{\sqrt{2}}$;
$\therefore \quad \theta = 45°$.

16. $6\cos^2\theta - \cos\theta - 1 = 0$;
$(3\cos\theta + 1)(2\cos\theta - 1) = 0$;
$\therefore \quad \cos\theta = \dfrac{1}{2}$ or $-\dfrac{1}{3}$;
$\therefore \quad \theta = 60°$.

17. $12\sin^2\theta - 4\sin\theta - 1 = 0$;
$(6\sin\theta + 1)(2\sin\theta - 1) = 0$;
$\therefore \quad \sin\theta = \dfrac{1}{2}$ or $-\dfrac{1}{6}$;
$\therefore \quad \theta = 30°$.

18. $2 - 2\cos^2\theta = 3\cos\theta$;
$2\cos^2\theta + 3\cos\theta - 2 = 0$;
$\therefore \quad (2\cos\theta - 1)(\cos\theta + 2) = 0$;
$\therefore \quad \cos\theta = \dfrac{1}{2}$, so that $\theta = 60°$.

19. $\tan\theta = 4 - \dfrac{3}{\tan\theta}$;

$\tan^2\theta - 4\tan\theta + 3 = 0$;

$\therefore (\tan\theta - 1)(\tan\theta - 3) = 0$;

$\therefore \tan\theta = 1$ or 3;

$\therefore \theta = 45°$.

20. $\cos^2\theta - 1 + \cos^2\theta = 2 - 5\cos\theta$;

$2\cos^2\theta + 5\cos\theta - 3 = 0$;

$\therefore (2\cos\theta - 1)(\cos\theta + 3) = 0$;

$\therefore \cos\theta = \dfrac{1}{2}$, so that $\theta = 60°$.

21. $\dfrac{1 + \tan^2\theta}{\tan\theta} = 2\sec\theta$;

$\dfrac{\sec^2\theta}{\tan\theta} = 2\sec\theta$;

$\therefore \sec\theta = 0$, or $\dfrac{\sec\theta}{\tan\theta} = 2$;

$\therefore \sin\theta = \dfrac{1}{2}$, so that $\theta = 30°$.

22. $\dfrac{4}{\sin\theta} + 2\sin\theta = 0$;

$2\sin^2\theta - 9\sin\theta + 4 = 0$;

$\therefore (2\sin\theta - 1)(\sin\theta - 4) = 0$;

$\therefore \sin\theta = \dfrac{1}{2}$, so that $\theta = 30°$.

23. $\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1}{\sin\theta}$;

$\sin^2\theta - \cos^2\theta = \cos\theta$;

$1 - 2\cos^2\theta = \cos\theta$;

$2\cos^2\theta + \cos\theta - 1 = 0$;

$\therefore (2\cos\theta - 1)(\cos\theta + 1) = 0$;

$\therefore \cos\theta = \dfrac{1}{2}$, or -1;

$\therefore \theta = 60°$.

24. $2\cos\theta + 2\sqrt{2} = \dfrac{3}{\cos\theta}$;

$2\cos^2\theta + 2\sqrt{2}\cos\theta - 3 = 0$;

$\therefore (\sqrt{2}\cos\theta - 1)(\sqrt{2}\cos\theta + 3) = 0$;

$\therefore \cos\theta = \dfrac{1}{\sqrt{2}}$;

$\therefore \theta = 45°$.

25. $\tan\theta(2\sin\theta - 1) = 2\sin\theta - 1$;

$(2\sin\theta - 1)(\tan\theta - 1) = 0$;

$\therefore \sin\theta = \dfrac{1}{2}$, or $\tan\theta = 1$;

$\therefore \theta = 30°$ or $45°$.

26. $6\dfrac{\sin\theta}{\cos\theta} - \dfrac{5\sqrt{3}}{\cos\theta} + 12\dfrac{\cos\theta}{\sin\theta} = 0$;

$6\sin^2\theta - 5\sqrt{3}\sin\theta + 12(1 - \sin^2\theta) = 0$;

$6\sin^2\theta + 5\sqrt{3}\sin\theta - 12 = 0$;

$\therefore (2\sin\theta - \sqrt{3})(3\sin\theta + 4\sqrt{3}) = 0$;

$\therefore \sin\theta = \dfrac{\sqrt{3}}{2}$, so that $\theta = 60°$.

27. $\tan\theta + \dfrac{3}{\tan\theta} = 4$;

$\tan^2\theta - 4\tan\theta + 3 = 0$;

$\therefore (\tan\theta - 1)(\tan\theta - 3) = 0$;

$\therefore \tan\theta = 1$ or 3.

28. $1 + \cot^2\theta + \cot^2\theta = 3\cot\theta$;

$2\cot^2\theta - 3\cot\theta + 1 = 0$;

$\therefore (2\cot\theta - 1)(\cot\theta - 1) = 0$;

$\therefore \cot\theta = 1$ or $\dfrac{1}{2}$.

MISCELLANEOUS EXAMPLES. A. PAGE 32.

3. If θ be the angle, we have $\sin\theta=\frac{21}{29}$, so that $\operatorname{cosec}\theta=\frac{29}{21}$.

Also $\cos\theta=\sqrt{1-\left(\frac{21}{29}\right)^2}=\frac{\sqrt{(29+21)(29-21)}}{29}=\frac{20}{29}$.

4. $\tan A=\frac{1}{\sqrt{\operatorname{cosec}^2 A-1}}=\frac{15}{\sqrt{2\times 32}}=\frac{15}{8}$;

$\sec A=\sqrt{1+\tan^2 A}=\frac{\sqrt{8^2+15^2}}{8}=\frac{17}{8}$.

5. First side $=\operatorname{cosec}^2 A-\cot^2 A-1=0$.

7. $b=\sqrt{a^2+c^2}=\sqrt{1681}=41$.

$$\cot A=\frac{c}{a}=\frac{9}{40};\quad \sec A=\frac{b}{c}=\frac{41}{9};\quad \sec C=\frac{b}{a}=\frac{41}{40}.$$

8. See Article 16.

9. First side $=\cos\theta(1-\cos\theta)\,\frac{1+\cos\theta}{\cos\theta}=1-\cos^2\theta=\sin^2\theta$.

10. We have $\sec^2\alpha=1+\tan^2\alpha=\frac{1+\cot^2\alpha}{\cot^2\alpha}$; $\therefore\ \sec\alpha=\frac{\sqrt{1+\cot^2\alpha}}{\cot\alpha}$.

Also $\operatorname{cosec}^2\alpha=1+\cot^2\alpha$; so that $\operatorname{cosec}\alpha=\sqrt{1+\cot^2\alpha}$.

11. First side $=3\left(\frac{1}{\sqrt{3}}\right)^2+\frac{1}{4}.2+5.1-\frac{2}{3}.\left(\frac{\sqrt{3}}{2}\right)^2=1+\frac{1}{2}+5-\frac{1}{2}=6$.

12. $\sin\alpha=\frac{1}{\sqrt{1+\cot^2\alpha}}=\frac{m}{\sqrt{m^2+n^2}}$; $\sec\alpha=\sqrt{1+\tan^2\alpha}=\frac{\sqrt{m^2+n^2}}{n}$.

13. m sexagesimal minutes $=\frac{m}{60\times 90}$ right angles,

n centesimal minutes $=\frac{n}{100\times 100}$ right angles.

$$\therefore\ \frac{m}{60\times 90}=\frac{n}{100\times 100};\ \text{whence } m=\cdot 54n.$$

14. $\cos A=\sqrt{1-\sin^2 A}=\sqrt{1-\frac{16}{25}}=+\frac{3}{5}$, since A is acute;

$$\therefore\ \tan A+\sec A=\frac{4}{3}+\frac{5}{3}=3.$$

15. First side $=\tan A\cot A\sin A\cot A=\cos A$.

16. $RQ=\sqrt{20^2+21^2}=\sqrt{841}=29$:

$$\therefore\ \tan Q=\frac{RP}{PQ}=\frac{20}{21};\quad \operatorname{cosec} Q=\frac{QR}{RP}=\frac{29}{20}.$$

17. First side $= \dfrac{\sin^2\alpha - \cos^2\alpha}{\sin\alpha\cos\alpha} \cdot \sin\alpha\cos\alpha = 1 - \cos^2\alpha - \cos^2\alpha = 1 - 2\cos^2\alpha.$

18. See Art. 39, Ex. 1.

19. Second side $= (\sqrt{3})^2 - 2 \cdot \left(\frac{1}{2}\right)^2 - \frac{3}{4}(\sqrt{2})^2 = 3 - \frac{1}{2} - \frac{3}{2} = 3 - 2$

$= \tan^2 60° - 2\tan^2 45°.$

20. (1) $3\sin\theta = 2\cos^2\theta;$
$2\sin^2\theta + 3\sin\theta - 2 = 0;$
$\therefore (2\sin\theta - 1)(\sin\theta + 2) = 0;$
$\therefore \sin\theta = \frac{1}{2}$, so that $\theta = 30°.$

(2) $5\cot\theta - \operatorname{cosec}^2\theta = 3;$
$5\cot\theta - 1 - \cot^2\theta = 3;$
$\cot^2\theta - 5\cot\theta + 4 = 0;$
$\therefore (\cot\theta - 1)(\cot\theta - 4) = 0;$
$\therefore \cot\theta = 1$, so that $\theta = 45°.$

21. First side $= 1 - (\sec^2 A - \tan^2 A)^2 = 1 - 1 = 0.$

22. $6\sin^2\theta - 11\sin\theta + 4 = 0;$ $\therefore (2\sin\theta - 1)(3\sin\theta - 4) = 0;$
$\therefore 2\sin\theta - 1 = 0;$ whence $\theta = 30°;$
or $3\sin\theta - 4 = 0;$ whence $\sin\theta = \frac{4}{3}$, which is impossible.

23. $\tan A + \tan B = \dfrac{a}{b} + \dfrac{b}{a} = \dfrac{a^2 + b^2}{ab} = \dfrac{c^2}{ab}.$

24. $c + c^{-1} = \cot A + \tan A = \dfrac{\cos^2 A + \sin^2 A}{\sin A\cos A} = \sec A\operatorname{cosec} A.$

25. First side $= \dfrac{\sqrt{2 - 2\cos A - 1 + 2\cos A - \cos^2 A}}{\cos A} = \dfrac{\sqrt{1 - \cos^2 A}}{\cos A} = \tan A.$

EXAMPLES. V. a. PAGE 37.

1. $c = \sqrt{a^2 - b^2} = \sqrt{16 - 12} = 2.$
$\sin C = \frac{c}{a} = \frac{1}{2};$ $\therefore C = 30°.$ $\sin B = \frac{b}{a} = \frac{\sqrt{3}}{2};$ $\therefore B = 60°.$

2. $a = \sqrt{b^2 - c^2} = \sqrt{144 - 36} = \sqrt{108} = 6\sqrt{3}.$
$\sin C = \frac{c}{b} = \frac{1}{2};$ $\therefore C = 30°;$ and $A = 90° - C = 60°.$

3. $c = \sqrt{a^2 + b^2} = \sqrt{144 + 48} = \sqrt{192} = 8\sqrt{3}.$
$\sin B = \frac{b}{c} = \frac{12}{8\sqrt{3}} = \frac{\sqrt{3}}{2};$ $\therefore B = 60°;$ and $A = 90° - B = 30°.$

4. $c = \sqrt{a^2 - b^2} = \sqrt{90 \times 30} = 30\sqrt{3}.$
$\sin C = \frac{c}{a} = \frac{\sqrt{3}}{2};$ $\therefore C = 60°,$ $B = 30°.$

5. $b=\sqrt{a^2+c^2}=\sqrt{800}=20\sqrt{2}$

$\sin C=\frac{c}{b}=\frac{1}{\sqrt{2}}$; $\therefore$ $C=45°$, $A=45°$.

6. $c=\sqrt{a^2+b^2}=\sqrt{75+3\times 75}=10\sqrt{3}$.

$\sin B=\frac{b}{c}=\frac{\sqrt{3}}{2}$; $\therefore$ $B=60°$, $A=30°$.

7. $b=c=2$; $\therefore$ $B=C=45°$.

$a=\sqrt{b^2+c^2}=2\sqrt{2}$.

8. $a=\sqrt{b^2-c^2}=\sqrt{3\times 36-27}=9$.

$\sin C=\frac{c}{b}=\frac{1}{2}$; $\therefore$ $C=30°$, $B=60°$.

9. $B=90°-A=60°$.

$\frac{b}{a}=\tan B$; $\therefore$ $b=9\sqrt{3}\,.\sqrt{3}=27$.

$\frac{c}{a}=\sec B$; $\therefore$ $c=9\sqrt{3}\,.2=18\sqrt{3}$.

10. $C=90°-30°=60°$.

$b=a\cos C=4\times\frac{1}{2}=2$;

$c=a\sin C=2\sqrt{3}$.

11. $B=90°-A=30°$.

$a=c\cos B=4\sqrt{3}$;

$b=c\sin B=4$.

12. $B=180°-C-A=90°$.

$a=b\cos C=3\sqrt{3}$;

$c=b\sin 30=3$.

13. $A=90°-C=30$.

$a=b\cos C=50$;

$c=b\sin C=50\sqrt{3}$.

14. $C=180°-A-B=90°$.

$a=b\tan A=20$; $c=b\sec A=40$.

15. $A=180°-B-C=90°$.

$b=c=4$; $a=\sqrt{b^2+c^2}=4\sqrt{2}$.

16. $A=180°-B-C=90°$.

$b=a\cos C=4$; $c=a\sin C=4\sqrt{3}$.

17. $a=b\tan A=\frac{49}{\cdot 07}=700$.

18. $a=c\sin A=50\times\cdot 62=31$.

19. $c=a\tan C=100\times\cdot 8647=86\cdot 47$.

20. $a=b\sec C=20\times 4\cdot 89=97\cdot 8$.

21. $C=90°-A=54°$.

$a=c\tan A=100\times\cdot 73=73$;

$b=c\sec A=100\times 1\cdot 24=124$.

22. $\sin C=\frac{c}{a}=\cdot 37$; $\therefore$ $C=21°\,43'$.

$B=90°-C=68°\,17'$;

$b=a\cos C=100\times\cdot 93=93$.

23. $C=90°-B=50°\,36'$.

$c=b\cot B=25\times 1\cdot 2174=30\cdot 435$;

$a=b\operatorname{cosec} B=25\times 1\cdot 5755=34\cdot 3875$.

24. $c=\sqrt{a^2+b^2}=\sqrt{124609}=353$.

$\tan B=\frac{b}{a}=\frac{272}{225}=1\cdot 209$;

$\therefore$ $B=50°\,24'$.

$A=90°-B=39°\,36'$.

EXAMPLES. V. b. PAGE 39.

1. $B=180°-30°-120°=30°=A$;
$\therefore\ CB=CA=20$;
$\therefore\ BD=BC\sin 60°=10\sqrt{3}$.

2. $c=BD\operatorname{cosec}30°=20$;
$a=BD\operatorname{cosec}45°=10\sqrt{2}$.

3. Since $B+C=90°$; $\therefore\ A=90°$.
$AB=BD\sec 30°=10\sqrt{3}$ ft.
$AC=AB\tan 30°=10$ ft.
$AD=AB\sin 30°=5\sqrt{3}$ ft.

4. Let QS be the perpendicular from Q on PR.
Then $PR=8\sec 60°=16$. $SR=8\cos 60°=4$. $\therefore\ SP=16-4=12$.

5. $SQ=36\tan 60°=36\sqrt{3}$. $RQ=36\tan 30°=12\sqrt{3}$.
$\therefore\ RS=36\sqrt{3}-12\sqrt{3}=24\sqrt{3}$.

6. We have $\angle PRQ=180°-135°=45°$; $\therefore\ \angle QPR=45°$;
$\therefore\ QR=QP=20$, and $\angle QPS=90°-30°=60°$,
$\therefore\ SQ=20\tan 60°=20\sqrt{3}$, $\therefore RS=20\sqrt{3}-20=20(\sqrt{3}-1)$.

7. Let AD be the perpendicular and let $AD=x$.
Then $\angle BAD=90°-45°=45°=\angle ABD$, $\therefore\ DA=DB=x$.
Now $\dfrac{DA}{DC}=\tan 60°$; $\therefore\ \dfrac{x}{x-40}=\sqrt{3}$; $\therefore\ x(\sqrt{3}-1)=40\sqrt{3}$.

$$\therefore\ x=\frac{40\sqrt{3}}{\sqrt{3}-1}=20\sqrt{3}(\sqrt{3}+1)=20(3+\sqrt{3}).$$

$\therefore$ perpendicular $=20(3+\sqrt{3})$.

8. Let $DC=x$.
Since $\angle DCB=90°-45°=45°=\angle CBD$; $\therefore\ DC=DB=x$.
And $\dfrac{DA}{DC}=\cot 32°.50'$; $\therefore\ \dfrac{x+59}{x}=1{\cdot}59$.

$$\therefore\ x+59=1{\cdot}59x;\quad \therefore\ x=100;$$

that is, $DC=DB=100$.

EXAMPLES. VI. a. PAGE 42.

1. See figure on page 40.
Let $BC=$height of chimney, $AC=300$ ft.,
then elevation $=\angle BAC=30°$, $\therefore\ BC=AC\tan 30°=100\sqrt{3}=173{\cdot}2$ ft.

2. See figure on page 40.
Let B be the top of the mast, $BC=160$ feet, A the boat observed.
Then $\angle BAC=30°$, $\angle ABC=60°$.
$\therefore$ distance required $=AC=BC\tan 60°=160\sqrt{3}=277{\cdot}12$ ft.

3. See figure on page 40.

Let BC represent the pole, and AC its shadow.

Then $\tan A = \frac{BC}{AC} = \frac{6}{2\sqrt{3}} = \sqrt{3}$; $\therefore$ angle of elevation $= 60°$.

4. See figure on page 40.

Let BC represent the tower; A the position of the observer.

Then $AC = 86{\cdot}6$ ft. and $\angle BAC = 30°$.

$\therefore$ height of tower $= AC \tan 30° = \frac{86{\cdot}6}{\sqrt{3}} = 50$ ft.

Distance $AB = BC \operatorname{cosec} 30° = 2BC = 100$ ft.

5. See figure on page 40.

Let AB represent the ladder, and BC the wall.

Then $AB = 45$ ft. $\angle ABC = 60°$.

$\therefore$ height of wall $= BC = AB \cos 60° = 22{\cdot}5$ ft.

Distance $AC = AB \sin 60° = \frac{45\sqrt{3}}{2} = 38{\cdot}97$ ft.

6. See figure on page 9.

Let DE, BC represent the masts, and OCE the horizon.

Then $\angle BOC = 33° 41'$. $BC = 40$ ft. $DE = 60$ ft.

And $OC = BC \cot 33° 41' = 60$ ft. $OE = DE \cot 33° 41' = 90$ ft.

$\therefore$ distance required $= OE - OC = 30$ ft.

7. See figure on page 40.

Let BC represent the cliff and A the observer.

Then $\angle BAC = 41° 18'$. $BC = 132$ yds.

$\therefore$ distance required $= AB = BC \operatorname{cosec} 41° 18' = \frac{132}{\cdot 66} = 200$ yds.

8. See figure on page 9.

Let BC, DE represent the chimneys, and O the observer.

Then $OC = 100$ yds. $\angle BOC = 27° 2'$.

Now $BC = OC \tan 27° 2' = 51$ yds.

$\therefore$ $DE = BC + 30$ yds. $= 81$ yds.

9. See figure on page 41.

Let PT be the tower, and Q, R the two points of observation.

Then $\angle PQR = 30°$, $\angle PRT = 60°$, $QR = 100$ yds.

$\therefore$ $\angle RPQ = 60° - 30° = 30° = \angle PQR$, $\therefore$ $RP = RQ = 100$ yds.

$\therefore$ height of tower $= RP \sin 60° = 50\sqrt{3} = 86{\cdot}6$ yds.

10. Let AB be the flagstaff, BC the building, D the point of observation.

Then $DC = 40$ ft. $\angle ADC = 60°$, $\angle BDC = 30°$;

$\therefore \angle DAB = 30° = \angle ADB$; $\therefore BA = BD = 40 \sec 30° = \dfrac{80}{\sqrt{3}} = 49{\cdot}19$ ft.

11. See figure on page 41.

Let PT be the spire, R, Q the two points of observation.

Then $QR = 200$ ft. $\angle PRT = 45°$, $\angle PQR = 30°$.

$\angle RPT = 45°$; $\therefore TP = TR$.

Let x ft. = height of spire.

Then $\dfrac{x}{x+200} = \tan 30° = \dfrac{1}{\sqrt{3}}$.

$\therefore x = \dfrac{200}{\sqrt{3}-1} = 100\,(1+\sqrt{3}) = 273{\cdot}2$ feet.

12. See figure on page 43.

Let CD represent the post, and AB the steeple.

Then $CD = 30$ ft. $\angle ACE = 30°$, $\angle ADB = 45°$;

$\therefore \angle DAB = 45° = \angle ADB$, $\therefore BA = BD = x$ feet, say.

$\therefore \tan 30° = \dfrac{AE}{EC} = \dfrac{x-30}{x}$; $\therefore x(\sqrt{3}-1) = 30\sqrt{3}$. $\therefore x = 70{\cdot}98$ ft.

That is height = distance = 70·98 ft.

13. Let B be the top of the hill, and C the point on the horizontal plane vertically below B. Let D be the position of the balloon when the observation is made. Draw DE perpendicular to BC.

Then $BC = 3300$ ft. $\angle BDE = 30°$, $\angle BAC = 60°$;

$\therefore AC = BC \cot 60° = 1100\sqrt{3}$ feet.

And $BE = DE \tan 30° = AC \tan 30° = 1100$ feet.

$\therefore DA = EC = 3300 - 1100 = 2200$ feet.

$\therefore$ the balloon rises 2200 feet in 5 minutes,

that is, $\dfrac{2200 \times 60}{1760 \times 3 \times 5}$ miles per hour, or 5 miles per hour.

14. See figure on page 44.

Let OA represent the monument, B, C the two objects, OP the horizontal line through O;

Then $\angle POC = 30°$; $\therefore \angle OCB = 30°$.

$\angle POB = 45°$; $\therefore \angle BOA = \angle OBA = 45°$; $\therefore AO = AB = 100$ feet.

Let x feet $= CB$ = required distance.

Then $\dfrac{100}{x+100} = \tan 30° = \dfrac{1}{\sqrt{3}}$; $\therefore x = 100\,(\sqrt{3}-1)$;

$\therefore$ distance required = 73·2 feet.

15. See figure on page 43.

Let AB represent the monument, CD the tower.

Then $AB = 96$ feet, and the angles are as in the figure;

$$\therefore DB = AB \cot 60° = 32\sqrt{3} \text{ feet};$$

$$\therefore AE = CE \tan 30° = DB \tan 30° = 32 \text{ feet}.$$

$$\therefore \text{height of tower} = CD = EB = 96 - 32 = 64 \text{ feet}.$$

16. See figure on page 44.

Let OA represent the cliff, B, C the two boats.

Then $OA = 150$ ft., $\angle OBA = 30°$, $\angle OCB = 15°$;

$$\therefore \angle BOC = 15°, \text{ and } BC = BO.$$

$$\therefore \text{required distance} = BC = BO = AO \operatorname{cosec} 30° = 300 \text{ ft}.$$

17. See figure on page 44.

Let O represent the top of the hill, B, C the milestones.

Then $\angle OBA = 45°$, $\angle OCB = 22°$; $\therefore \angle BOA = 45°$, so that $AO = AB$.

Let $AO = AB = x$ yards.

Then $$\frac{AC}{AO} = \frac{x + 1760}{x} = \cot 22° = 2{\cdot}475; \quad \therefore 1{\cdot}475x = 1760.$$

$$\therefore \text{height of hill} = x = 1193 \text{ yds. nearly}.$$

18. See figure on page 44.

Let OA represent the lighthouse, B, C the two rocks.

Then $OA = 80$ yds., $\angle OBA = 75°$, $\angle OCB = 15°$, $\angle COA = 75°$;

$$\therefore AB = OA \cot 75° = 80 \times {\cdot}268 \text{ yds}.$$

Let $CB = x$ yds.

Then $x + 80 \times {\cdot}268 = OA \cot 15° = 80 \times 3{\cdot}732$, $\therefore x = 80 \times 3{\cdot}464 = 277{\cdot}12$ yds.

$$\therefore \text{required distance} = 277{\cdot}12 \text{ yds}.$$

EXAMPLES. VI b. PAGE 47.

1. Let A, B be the two positions of the observer, P, Q the two objects. Then $AB = 800$ yds., and PQA is a straight line making $\angle PAB$ equal to 45°. Also $\angle PBA = 90°$, $\angle QBA = 45°$. $\therefore QA = QB = QP$.

And $QA = AB \cos 45° = \dfrac{800}{\sqrt{2}} = 565{\cdot}6$ yds. $PA = 2QA = 1131{\cdot}2$ yds.

Thus the required distances are 565·6 yds., 1131·2 yds.

2. Let A, B be the two positions of the observer, P, Q the two ships; then APQ is a straight line at right angles to AB.

And $AB = 3$ miles, $\angle ABP = 30°$, $\angle ABQ = 60°$.

$$\therefore BP = AB \sec 30° = \frac{6}{\sqrt{3}} = 3{\cdot}464 \text{ miles}, \quad BQ = AB \sec 60° = 6 \text{ miles}.$$

Thus the required distances are 3·464 miles, 6 miles.

3. Let O represent the harbour and ON, OE, OS, OW the directions of North, South, East, West.

Let P, Q be the positions of the two ships at 2 p.m.

Then $\angle POW = 28°$, $\angle QOE = 62°$; $\therefore \angle POQ = 90°$.

Also $OP = 2 \times 10 = 20$ miles. $OQ = 2 \times 10\frac{1}{2} = 21$ miles.

$\therefore$ distance $= PQ = \sqrt{20^2 + 21^2} = 29$ miles.

4. Let O be the position of the lighthouse, and P, Q the points at which the steamer enters and leaves the light.

Then PQ lies East and West, and OP, OQ are the directions of N.E., N.W.

$\therefore \angle POQ = 90°$, $\angle PQO = \angle QPO = 45°$, $OP = OQ = 5$ miles.

$\therefore PQ = \sqrt{25 + 25} = 5\sqrt{2}$ miles. $\therefore$ steamer sails $5\sqrt{2}$ miles in $30\sqrt{2}$ minutes, that is, the speed of steamer is 10 miles per hour.

5. Let O, P, Q be the first positions of the ship and lighthouses. OA the direction in which the ship is sailing, A its second position.

Then $OA = 10$ miles, $\angle OAP = 45°$, $\angle AOP = 90°$, $\angle PAQ = 22\frac{1}{2}°$.

$\therefore \angle PQA = 45° - 22\frac{1}{2}° = 22\frac{1}{2}° = PAQ$; $\therefore PA = PQ$.

And $OP = OA = 10$ miles, $PA = OA \sec 45° = 10\sqrt{2}$ miles.

$\therefore OQ = OP + PQ = OP + PA = 10(\sqrt{2} + 1) = 24{\cdot}14$ miles.

$\therefore$ distances are 10 miles, 24·14 miles.

6. As before let O be the port, and ON, OE, OS, OW the directions of the cardinal points of the compass.

Let P, Q be the positions of the ships at the end of an hour.

Then $OP = 8$ miles, $OQ = 8\sqrt{3}$ miles.

And $\angle PON = 35°$, $\angle QOS = 55°$; $\therefore \angle POQ = 90°$.

$\therefore$ distances apart $= PQ = \sqrt{64 + 3 \times 64} = 16$ miles.

Also $\tan QPO = \frac{8\sqrt{3}}{8} = \sqrt{3}$; $\therefore \angle QPO = 60°$.

$\therefore \angle QPO - \angle PON = 60° - 35° = 25°$;

$\therefore$ bearing of the second vessel as observed from the first is S. 25° W.

7. Let A be the lighthouse, O, P the two positions of the vessel.

Then AP the direction of S., AO the direction of E.S.E., OP the direction of S.S.W.

$\therefore \angle AOP = 90°$; $\angle PAO = 90° - 22\frac{1}{2}° = 67\frac{1}{2}°$; and $AO = 4$ miles.

$\therefore PO = AO \tan 67\frac{1}{2}° = 4 \times 2{\cdot}414 = 9{\cdot}656$ miles.

$\therefore$ the vessel sails at the rate of 9·656 miles per hour.

8. We have $\angle CAB = 10° + 50° = 60°$, $\angle ABC = 180° - 50° - 40° = 90°$,

$BC = 10$ miles.

$\therefore AB = BC \cot 60° = \frac{10}{\sqrt{3}} = 5{\cdot}77$ mls.; $AC = BC \operatorname{cosec} 60 = \frac{20}{\sqrt{3}} = 11{\cdot}54$ mls.

9. Let O be the lighthouse, and A, B the two positions of the ship.

Then $\angle OAB = \angle OBA = 45°$; $OA = OB = 15$ miles.

$\therefore AB = 15\sqrt{2}$ miles $= \frac{15\sqrt{2} \times 60}{69}$ knots.

$\therefore$ the ship in $1\frac{1}{2}$ hours sails $\frac{15\sqrt{2} \times 60}{69}$ knots.

$\therefore$ in a day it sails $\frac{15\sqrt{2} \times 60}{69} \times \frac{24 \times 2}{3}$ knots, that is, 295·09 knots.

10. Let O be the lighthouse, A, B the two positions of the coaster.

Then AB is in direction S.E., and OA is in direction N.E.; $\therefore \angle OAB = 90°$.

Also $\angle AOB = 45° + 15° = 60°$, and $OA = 9$ miles;

$\therefore AB = OA \tan 60° = 9\sqrt{3}$ miles; $\therefore$ coaster sails $9\sqrt{3}$ miles in 3 hours.

$\therefore$ rate of the coaster's sailing $= 5{\cdot}196$ miles per hour.

Also $OB = AO \sec 60° = 18$ miles.

That is, the distance of the coaster from the lighthouse at time of second observation $= 18$ miles.

11. Let P be the position of the vessel when it is N.E. of A and N.W. of B. Then $\angle APB = 90°$.

Also $\angle PAB = 45° - 15° = 30°$. $\therefore PA = AB \cos 30° = 6\sqrt{3}$ miles.

Now the direction S. 15° E. is at right angles to the direction E. 15° N.

$\therefore$ the ship crosses AB at right angles. Draw PN perpendicular to AB.

Then $PN = AP \sin 30° = 3\sqrt{3}$ miles; therefore the ship will reach N in $\frac{3\sqrt{3}}{10}$ hours, that is in 31·176 minutes.

$\therefore$ the ship will cross the line at about 31′ past midnight.

12. Let P, Q be the two spires.

Then $\angle PAB = 90°$, and $\angle PBQ = 37\frac{1}{2}° - 7\frac{1}{2}° = 30°$;

$\therefore \angle QBA = 90° - 37\frac{1}{2}° - 22\frac{1}{2}° = 30°$;

$\therefore \angle BPQ = 30° = \angle PBQ$; so that $QB = QP = 1{\cdot}5$ miles.

$\therefore AB = BQ \cos 30° = \frac{3\sqrt{3}}{4}$ miles.

$\therefore$ the train travels $\frac{3\sqrt{3}}{4}$ miles in 2 minutes,

that is, $\frac{3\sqrt{3}}{4} \times 30$ miles per hour, or 38·97 miles per hour.

EXAMPLES. VII. a. PAGE 54.

For Examples 1—22, see Art. 64; the following solutions will suffice as illustrations.

6. Radian measure of $57\frac{1}{2}$ degrees $= \frac{57\frac{1}{2}}{180}\pi = \frac{23\pi}{72}$.

7. Radian measure of $14\frac{2}{5}$ degrees $= \frac{14\frac{2}{5}}{180}\pi = \frac{2\pi}{25}$.

10. Radian measure of $37\frac{1}{2}$ degrees $= \frac{37\frac{1}{2}}{180} \times 3{\cdot}1416 = {\cdot}6545$.

11. Radian measure of $68\frac{3}{4}$ degrees $= \frac{68\frac{3}{4}}{180} \times 3{\cdot}1416$

$$= \frac{275}{4 \times 180} \times 3{\cdot}1416 = \frac{55}{4 \times 3} \times {\cdot}2618 = 1{\cdot}1999.$$

16. $\frac{7\pi}{45}$ radians $= \frac{7 \times 180}{45}$ degrees $= 28°$.

19. ${\cdot}3927\rho = {\cdot}3927 \times \frac{180}{\pi}$ [Art. 63]

$$= \frac{3927}{31416} \times 180 = \frac{180}{8} = 22°\,30'.$$

22. $2{\cdot}8798\rho = 2{\cdot}8798 \times \frac{180}{\pi} = \frac{28798}{31416} \times 180 = \frac{11}{12} \times 180 = 165°$.

23. Here $\frac{\theta}{\pi} = \frac{36{\cdot}54}{180} = \frac{2{\cdot}03}{10}$;

$\therefore\ \theta = {\cdot}203 \times \frac{22}{7} = {\cdot}638$.

$60)\,24$
$60)\,32{\cdot}4$
$\cdot 54$

25. Here $\frac{\theta}{\pi} = \frac{116{\cdot}046}{180} = {\cdot}6447$;

$\therefore\ \theta = {\cdot}6447 \times \frac{22}{7} = 2{\cdot}0262$.

$60)\,45{\cdot}6$
$60)\,2{\cdot}76$
$\cdot 046$

27. A radian $= \frac{180}{\pi}$ degrees;

$\therefore$ no. of seconds in a radian $= \frac{180 \times 60 \times 60}{\pi}$

$$= 180 \times 60 \times 60 \times {\cdot}31831$$
$$= 206265 \text{ nearly.}$$

28. Since $1° = \frac{\pi}{180}$ radians, the radian measure of

$$1'' = \frac{3{\cdot}1416}{180 \times 60 \times 60} = {\cdot}0000048.$$

EXAMPLES. VII. b. PAGE 56.

5. $\cot^2\frac{\pi}{6}+4\cos^2\frac{\pi}{4}+3\sec^2\frac{\pi}{6}=(\sqrt{3})^2+4\left(\frac{1}{\sqrt{2}}\right)^2+3\left(\frac{2}{\sqrt{3}}\right)^2=3+2+4=9.$

6. $3\tan^2\frac{\pi}{6}-\frac{1}{3}\sin^2\frac{\pi}{3}-\frac{1}{2}\operatorname{cosec}^2\frac{\pi}{4}+\frac{4}{3}\cos^2\frac{\pi}{6}$

$$=3\left(\frac{1}{\sqrt{3}}\right)^2-\frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^2-\frac{1}{2}(\sqrt{2})^2+\frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^2$$

$$=1-\frac{1}{4}-1+1=\frac{3}{4}.$$

7. $\left(\sin\frac{\pi}{6}+\cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{3}-\cos\frac{\pi}{3}\right)\sec\frac{\pi}{3}$

$$=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)2=2\left(\frac{3}{4}-\frac{1}{4}\right)=1.$$

8. First side $=\sin\theta\operatorname{cosec}\theta-\cot\theta\tan\theta=1-1=0.$

9. First side $=\dfrac{\cos^2\theta}{\sin\theta}-\cot^2\theta\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}-\dfrac{\cos^2\theta}{\sin\theta}=0.$

10. First side $=\dfrac{\cos^2\theta}{\operatorname{cosec}\theta}\cdot\dfrac{\sec\theta}{\tan\theta}=\cos^2\theta\sin\theta\times\dfrac{1}{\cos\theta}\cdot\dfrac{\cos\theta}{\sin\theta}=\cos^2\theta.$

11. First side $=\tan\theta+\cot\theta=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$

$$=\frac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta}=\frac{1}{\cos\theta\sin\theta}=\sec\theta\operatorname{cosec}\theta=\sec\theta\sec\left(\frac{\pi}{2}-\theta\right).$$

12. First side $=\sec^2\theta+\operatorname{cosec}^2\theta=\sec^2\theta(1+\cot^2\theta)$

$$=\sec^2\theta\operatorname{cosec}^2\theta=(1+\tan^2\theta)\sec^2\left(\frac{\pi}{2}-\theta\right).$$

14. See Art. 69.

15. The second side $=\dfrac{1}{\cos^2\frac{\pi}{3}}-\dfrac{1}{\cos^2\frac{\pi}{6}}=4-\frac{4}{3}=\frac{8}{3}$

$$=3-\frac{1}{3}=\tan^2\frac{\pi}{3}-\cot^2\frac{\pi}{3}.$$

16. The expression $=(\sin\theta+\cos\theta)^2+(\cos\theta-\sin\theta)^2$

$$=2(\sin^2\theta+\cos^2\theta)=2.$$

EXAMPLES. VII. c. PAGE 60.

1. Here $\dfrac{\text{arc}}{\text{radius}}=\dfrac{1{\cdot}6}{8}=\dfrac{1}{5}=$ radian measure required.

2. Here $r=\frac{a}{\theta}=\frac{219}{\cdot73}=300$ ft. 3. Radian measure $=\frac{7\cdot5}{7\cdot5}=1$.

4. Here $a=r\theta=1\cdot625\times3\cdot6=5\cdot85$ yds.

5. Here $a=627$ inches; $\theta=1\cdot9$. $\therefore\ r=\frac{a}{\theta}=\frac{627}{1\cdot9}=330$ inches.

6. Each revolution $=2\pi$ radians;

$$\therefore\ 5 \text{ radians}=\frac{5}{2\pi}=\frac{7\times5}{44} \text{ revolutions.}$$

And each revolution takes $\frac{1}{35}$ of a second;

$$\therefore \text{ required time}=\frac{1}{35}\times\frac{35}{44}=\frac{1}{44} \text{ of a second.}$$

7. Here $a=r\theta$, where $r=28$ inches and $\theta=\frac{1}{3}\times\frac{44}{7}$.

$$\therefore\ a=\frac{44}{21}\times28=58\tfrac{2}{3} \text{ inches.}$$

8. Radian measure of $75°=\frac{75\times3\cdot1416}{180}$.

If r be the length of the rope in yards, we have

$$r=\frac{52\cdot36\times180}{3\cdot1416\times75}=40 \text{ yards.}$$

9. Here $a=r\theta$, where $r=3960$ miles, and $\theta=$ radian measure of 1 minute;

$$\therefore\ a=3960\times\frac{3\cdot1416}{180\times60}=1\cdot15192 \text{ miles.}$$

10. The number of radians in the angle $=\frac{11}{2}\times\frac{1}{1760\times12\times3}$;

$$\therefore \text{ number of seconds}=\frac{11}{2}\times\frac{1}{1760\times12\times3}\times\frac{180}{3\cdot1416}\times60\times60$$
$$=17\cdot904, \text{ on reduction.}$$

11. With the figure on p. 59, we have to find the angle POQ when $=3960$ miles, and $PQ=145\cdot2$.

$$\therefore \text{ radian measure}=\frac{145\cdot2}{3960};$$

$$\therefore \text{ no. of degrees}=\frac{145\cdot2}{3960}\times\frac{180\times7}{22}=\frac{1452}{220}\times\frac{7}{22}=\frac{66\times7}{220}=\frac{21}{10}.$$

$$\therefore \text{ the angle}=2°\,6'.$$

12. Here $r=\frac{a}{\theta}$, where $a=1$ foot, and θ is the radian measure of $\frac{14}{11}$ degrees.

$$\therefore\ r=1\div\left(\frac{14}{11}\times\frac{\pi}{180}\right)=\frac{11}{14}\times\frac{180\times7}{22}=45 \text{ ft.}$$

MISCELLANEOUS EXAMPLES. B. PAGE 61.

1. $D=\frac{180}{\pi}\times\cdot15708=\frac{180\times15708}{314160}=9.$

2. See figure on p. 14. $b=c\cos A=\frac{110\sqrt{3}}{2}=55\times1\cdot732=95\cdot26.$

3. If $\frac{12\pi}{25}$ is represented by $\frac{8}{5}$,

π ..... $\frac{8}{5}\times\frac{25}{12}$, or $\frac{10}{3}$.

$\therefore 180\div\frac{10}{3}$ is the number of degrees in the unit angle; that is, the unit is 54°.

4. Here $r=\frac{a}{\theta}$, where $a=1$ inch and θ is the radian measure of 1′.

$\therefore r=1\div\left(\frac{\pi}{180}\times\frac{1}{60}\right)=180\times60\times\frac{1}{\pi}=180\times60\times\cdot31831=3438$ inches.

5. (1) First side $=(\sin\alpha+\cos\alpha)\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}\right)$

$$=\frac{(\sin\alpha+\cos\alpha)(\sin^2\alpha+\cos^2\alpha)}{\sin\alpha\cos\alpha}=\sec\alpha+\operatorname{cosec}\alpha.$$

(2) First side $=(\sqrt{3}+1)(3-\sqrt{3})=\sqrt{3}(\sqrt{3}+1)(\sqrt{3}-1)$

$$=3\sqrt{3}-\sqrt{3}=\tan^3 60°-2\sin 60°.$$

6. In the figure on p. 14, if BC represents the chimney and AC the shadow we have

$$\tan A=\frac{60}{3\times20\sqrt{3}}=\frac{3}{3\sqrt{3}}=\frac{1}{\sqrt{3}};\quad\therefore A=30°.$$

7. (1) First side $=2\tan^2\theta+5\tan\theta+2$

$$=2(1+\tan^2\theta)+5\tan\theta=2\sec^2\theta+5\tan\theta.$$

(2) $$\frac{\cot^2\alpha}{1+\operatorname{cosec}\alpha}=\frac{\operatorname{cosec}^2\alpha-1}{\operatorname{cosec}\alpha+1}=\operatorname{cosec}\alpha-1.$$

8. Expressed in radians the third angle $=\pi-\left(\frac{\pi}{4}+\frac{5\pi}{8}\right)=\frac{\pi}{8}$. The sexagesimal equivalent is $22\frac{1}{2}°$.

9. Let x be the number of degrees in the angle; then

$$x=14\left(\frac{\pi}{180}x\right)+51,\text{ or }x-\frac{11}{45}x=51;\text{ whence }x=67\tfrac{1}{2}.$$

10. With the figure of Art. 45, we have

$$a = b \tan 60° = 6\sqrt{3}\,; \quad c = b \sec 60° = 6 \times 2 = 12.$$

Also the perpendicular from C on $AB = b \sin 60° = 3\sqrt{3}$.

11. (1) First side $= \cot\theta + \tan\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} = \operatorname{cosec}\theta \sec\theta = \operatorname{cosec}\theta \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right).$$

(2) First side $= \operatorname{cosec}^2\theta + \sec^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$ [Art. 31, Ex. 1.]

$$= \operatorname{cosec}^2\theta \operatorname{cosec}^2\left(\frac{\pi}{2} - \theta\right).$$

12. In the figure on p. 41, let PT be the pillar; then

$QR = 20$ ft., $\angle PQR = 30°$, $\angle PRT = 60°$, and $PR = QR = 20$ ft.

$$\therefore\ PT = PR \sin 60° = \frac{20\sqrt{3}}{2} = 10\sqrt{3} = 17{\cdot}32 \text{ ft.}$$

13. First side $= \sin^2 A \left(\dfrac{1}{\cos^2 A} - 1\right) = \dfrac{\sin^2 A\,(1 - \cos^2 A)}{\cos^2 A}$

$$= \sin^2 A \,.\, \sin^2 A \,.\, \sec^2 A = \sin^4 A \sec^2 A.$$

14. x grades $= \dfrac{9x}{10}$ degrees;

$$\text{also } \frac{\pi x}{300} \text{ radians} = \frac{\pi x}{300} \times \frac{180}{\pi} = \frac{3x}{5} \text{ degrees};$$

$\therefore\ 3x + \dfrac{9x}{10} + \dfrac{3x}{5} = 180$; whence $x = 40$. Thus the angles are 120°, 36°, 24°.

15. Expression $= \left(\dfrac{\sqrt{3}}{2}\right)^3 \sqrt{3} - 2\,(\sqrt{2})^2 + 3 \,.\, \dfrac{1}{2} \,.\, 1 - (\sqrt{3})^2$

$$= \frac{9}{8} - 4 + \frac{3}{2} - 3 = -\frac{35}{8}.$$

16. (1) First side $= 1 + \tan^2 A + 2\tan A + 1 + \cot^2 A + 2\cot A$

$$= \sec^2 A + \operatorname{cosec}^2 A + 2 \,.\, \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$$

$$= \sec^2 A + \operatorname{cosec}^2 A + 2 \sec A \operatorname{cosec} A = (\sec A + \operatorname{cosec} A)^2.$$

(2) First side $= (\sec\alpha - 1)^2 - \sin^2\alpha\,(\sec\alpha - 1)^2 = (\sec\alpha - 1)^2\,(1 - \sin^2\alpha)$

$$= (\sec\alpha - 1)^2 \cos^2\alpha = (1 - \cos\alpha)^2.$$

17. (1) $a^2 + b^2 > 2ab$, since $(a - b)^2$ is positive.

Therefore $\dfrac{a^2 + b^2}{2ab} > 1$. Hence $\operatorname{cosec}\theta = \dfrac{a^2 + b^2}{2ab}$ is possible.

(2) $a^2+1>2a$: so that $a+\frac{1}{a}>2$.

Hence $2\sin\theta = a+\frac{1}{a}$ is impossible [Art. 16].

18. The height of the balloon $= 660\tan 60°$ ft. $= 660\sqrt{3}$ feet.

$\therefore$ the balloon rises $660\times\sqrt{3}$ feet in 1·5 minutes.

$\therefore$ it rises $\frac{660\times\sqrt{3}\times 60}{1\cdot5\times 3\times 1760}$, or 8·66 miles per hour.

19. Let $x°$ be the common difference between the angles, then they are $36°,\ 36°+x°,\ 36°+2x°$,

$\therefore\ 3x+3\times 36=180$; whence $x=24$.

$\therefore$ the angles are $36°, 60°, 84°$, or $\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15}$ radians.

20. First side $=\sin^2\alpha\,(1+\tan^2\beta)+\tan^2\beta\,(1-\sin^2\alpha)=\sin^2\alpha+\tan^2\beta$.

21. Let $CD=x=$ the perpendicular.

Then $\angle CBD = 180°-116°\,33'=63°\,27'$; $\therefore\ x=DB\tan 63°\,27'=2DB$.

And $x=DA\tan 42°=\left(\frac{x}{2}+55\right)\times\cdot9$; whence $x=4\cdot5\times 20=90$.

22. (1) First side $=\frac{\cos\alpha\,(1+\cos\alpha)+\sin^2\alpha}{\sin\alpha\,(1+\cos\alpha)}=\frac{\cos\alpha+1}{\sin\alpha\,(1+\cos\alpha)}=\operatorname{cosec}\alpha$.

(2) First side $=\frac{1-\cos\alpha}{\sin\alpha}\left(\frac{1}{\cos\alpha}-\cos\alpha\right)=(1-\cos\alpha)\frac{\sin\alpha}{\cos\alpha}=\tan\alpha-\sin\alpha$.

23. First side $=\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)^2=\frac{2+\sqrt{3}}{2-\sqrt{3}}=\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}=\frac{1+\cos 30°}{1-\cos 30°}$.

24. Let the man start from A and walk to B, and let C denote the position of the windmill.

Then we have $\angle ACB=90°$, $\angle CAB=30°$, $BC=1$ mile.

$\therefore\ AB=BC\operatorname{cosec}30=2$ miles. $AC=BC\tan 60°=1\cdot732$ miles.

And rate of walking is 2 miles per half hour, or 4 miles an hour.

25. The complement of $\frac{3\pi}{8}=\frac{\pi}{2}-\frac{3\pi}{8}=\frac{\pi}{8}$ radians.

26. (1) $3\sin\theta+4-4\sin^2\theta=\frac{9}{2}$, (2) $\tan\theta+\frac{2}{\sqrt{3}}=\frac{1}{\tan\theta}$,

$8\sin^2\theta-6\sin\theta+1=0$; $\sqrt{3}\tan^2\theta+2\tan\theta-\sqrt{3}=0$;

$\therefore (4\sin\theta-1)(2\sin\theta-1)=0$; $\therefore (\sqrt{3}\tan\theta-1)(\tan\theta+\sqrt{3})=0$;

$\therefore \sin\theta=\frac{1}{4}$, or $\frac{1}{2}$, $\therefore \tan\theta=\frac{1}{\sqrt{3}}$, or $-\sqrt{3}$.

$\therefore \theta=30°$. $\therefore \theta=30°$.

27. Here $\frac{5\sin\alpha-3\cos\alpha}{\sin\alpha+2\cos\alpha}=\frac{5\tan\alpha-3}{\tan\alpha+2}=\frac{20-15}{4+10}=\frac{5}{14}$.

28. First side $=\dfrac{1-\sin A\cos A}{\cos A\,.\,\dfrac{\sin A-\cos A}{\sin A\cos A}}\times\dfrac{\sin A-\cos A}{\sin^2 A-\sin A\cos A+\cos^2 A}$

$=\dfrac{\sin A\,(1-\sin A\cos A)}{1-\sin A\cos A}=\sin A$.

29. The distance $=195{\cdot}2\operatorname{cosec}77°\,26'$ yds. $=\frac{195{\cdot}2}{{\cdot}976}=200$ yds.

30. We have $70°=\frac{7\pi}{18}$ radians.

$\therefore$ distance required $=27\times\frac{7\pi}{18}$ feet $=\frac{21}{2}\times\frac{22}{7}=33$ feet.

EXAMPLES. VIII. a. PAGE 70.

18. $\sin 420°=\sin(360°+60°)=\sin 60°=\frac{\sqrt{3}}{2}$.

20. $\tan(-315°)=\tan(-360°+45°)=\tan 45°=1$.

22. $\operatorname{cosec}(-330°)=\operatorname{cosec}(-360°+30°)=\operatorname{cosec}30°=2$.

24. $\cot\frac{17\pi}{4}=\cot\left(4\pi+\frac{\pi}{4}\right)=\cot\frac{\pi}{4}=1$.

26. $\tan\left(-\frac{5\pi}{3}\right)=\tan\left(-2\pi+\frac{\pi}{3}\right)=\tan\frac{\pi}{3}=\sqrt{3}$.

EXAMPLES. VIII. b. PAGE 72.

1. The boundary line of 120° lies in the second quadrant,

$\therefore \cos 120°=-\sqrt{1-\sin^2 120°}=-\sqrt{1-\frac{3}{4}}=-\frac{1}{2}$;

$\therefore \tan 120°=\frac{\sin 120°}{\cos 120°}=-\sqrt{3}$.

2. The boundary line of 135° lies in the second quadrant;

$$\therefore \sec 135° = -\sqrt{1+\tan^2 135°} = -\sqrt{2};$$

and $$\sin 135° \sec 135° = -1; \quad \therefore \sin 135° = \frac{1}{\sqrt{2}}.$$

3. The boundary line of 240° lies in the third quadrant;

$$\therefore \sec 240° = -\sqrt{1+\tan^2 240°} = -2; \quad \therefore \cos 240° = -\frac{1}{2}.$$

4. The boundary line of 202° 37′ lies in the third quadrant;

$$\therefore \cos A = -\sqrt{1-\sin^2 A} = -\sqrt{1-\frac{25}{169}} = -\frac{12}{13}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{12}{5}.$$

5. The boundary line of 143° 8′ lies in the second quadrant; and

$$\operatorname{cosec} A = 1\tfrac{2}{3}; \quad \therefore \sin A = \frac{3}{5}; \quad \therefore \cos A = -\sqrt{1-\sin^2 A} = -\frac{4}{5};$$

$$\therefore \sec A = -\frac{5}{4}. \quad \text{Hence } \tan A = \frac{\sin A}{\cos A} = -\frac{3}{4}.$$

6. The boundary line of 216° 52′ lies in the third quadrant;

$$\therefore \sin A = -\sqrt{1-\cos^2 A} = -\sqrt{1-\frac{16}{25}} = -\frac{3}{5}. \quad \text{Also } \cot A = \frac{\cos A}{\sin A} = \frac{4}{3}.$$

7. The boundary line of $\frac{2\pi}{3}$ lies in the second quadrant;

and $$\sec \frac{2\pi}{3} = -2; \quad \therefore \cos \frac{2\pi}{3} = -\frac{1}{2}. \quad \therefore \sin \frac{2\pi}{3} = +\sqrt{1-\cos^2 \frac{2\pi}{3}} = \frac{\sqrt{3}}{2}.$$

$$\cot \frac{2\pi}{3} = \frac{\cos \frac{2\pi}{3}}{\sin \frac{2\pi}{3}} = -\frac{1}{\sqrt{3}}.$$

8. The boundary line of $\frac{5\pi}{4}$ lies in the third quadrant;

$$\therefore \cos \frac{5\pi}{4} = -\sqrt{1-\sin^2 \frac{5\pi}{4}} = -\frac{1}{\sqrt{2}};$$

$$\therefore \sec \frac{5\pi}{4} = -\sqrt{2}, \quad \text{and } \tan \frac{5\pi}{4} = \frac{\sin \frac{5\pi}{4}}{\cos \frac{5\pi}{4}} = 1.$$

9. We have $$\sin A = \pm\sqrt{1-\cos^2 A} = \pm\sqrt{1-\frac{144}{169}} = \pm\frac{5}{13}.$$

$$\therefore \tan A = \frac{\sin A}{\cos A} = \pm\frac{5}{12}.$$

EXAMPLES. IX. PAGE 79.

6. Expression $= 1 . (-1)^2 - 2(-1) . 1 = 1 + 2 = 3$.

7. Expression $= 3 . 0 . (-1) + 2 . 1 - 1 = 2 - 1 = 1$.

8. Expression $= 2 . (-1)^2 . 1 + 3(-1)^3 - 1 = 2 - 3 - 1 = -2$.

9. Expression $= 0 \times 0 + 1 - (-1) = 1 + 1 = 2$.

MISCELLANEOUS EXAMPLES. C. PAGE 81.

1. If $\tan A = -\frac{3}{4}$, the boundary of A will lie either in second or fourth quadrant. [See figure on page 72.]

In either position the radius vector $= \sqrt{3^2 + 4^2} = 5$.

Hence $\cos XOP = -\frac{4}{5}$; $\cos XOP' = \frac{4}{5}$.

2. First side $= (2 + \sin A)(1 - 2 \sin A) \sec A$

$= (2 - 3 \sin A - 2 \sin^2 A) \sec A$

$= (2 \cos^2 A - 3 \sin A) \sec A = 2 \cos A - 3 \tan A$.

3. $a = \sqrt{c^2 - b^2} = \sqrt{(21)^2 - (10{\cdot}5)^2} = 21 \sqrt{1 - \frac{1}{4}} = \frac{21\sqrt{3}}{2}$.

$\sin A = \frac{a}{c} = \frac{\sqrt{3}}{2}$; whence $A = 60°$, $B = 30°$.

4. A lies between 180° and 270°;

$\therefore \tan A = +\sqrt{\sec^2 A - 1} = \sqrt{\left(\frac{25}{7}\right)^2 - 1} = \frac{24}{7}$; and $\cot A = \frac{7}{24}$.

5. We have $19° = \frac{19\pi}{180}$ radians.

Also the radius of the earth $= 3960$ miles.

$\therefore$ required distance $= \frac{19\pi}{180} \times 3960 = 19 \times 22 \times \pi = 1313$ miles nearly.

6. Let AB represent the cliff and P, Q the positions of the two boats.

Then $AB = 200$ ft. $\angle APB = 34° 30'$, $\angle AQB = 18° 40'$;

$\therefore QB = AB \cot 18° 40' = 200 \times 2{\cdot}96 = 592$ ft.

$PB = AB \cot 34° 30' = 200 \times 1{\cdot}455 = 291$ ft.

$\therefore$ required distance $= QB - PB = 301$ ft.

7. Since the boundary line of A is in the third quadrant

$$\therefore \sec A = -\sqrt{1+\tan^2 A} = -\sqrt{1+\frac{16}{9}} = -\frac{5}{3},$$

$$\therefore \cos A = -\frac{3}{5}, \text{ and } \sin A = -\frac{4}{5}.$$

$$\therefore 2\cot A - 5\cos A + \sin A = 2\cdot\frac{3}{4} - 5\left(-\frac{3}{5}\right) - \frac{4}{5} = \frac{3}{2} + 3 - \frac{4}{5} = 3\tfrac{7}{10}.$$

8. We have $71°\,36'\,3{\cdot}6'' = 71{\cdot}601° = \dfrac{71{\cdot}601\pi}{180}$ radians.

$$\therefore \text{required radius} = 15 \div \frac{71{\cdot}601\pi}{180} = \frac{15\times 180}{71{\cdot}601\pi} = 12{\cdot}003 \text{ inches}.$$

9. $$\text{First side} = \frac{\tan^3\theta}{\sec^2\theta} + \frac{\cot^3\theta}{\operatorname{cosec}^2\theta}$$

$$= \frac{\sin^3\theta}{\cos\theta} + \frac{\cos^3\theta}{\sin\theta} = \frac{\sin^4\theta + \cos^4\theta}{\sin\theta\cos\theta}$$

$$= \frac{(\sin^2\theta+\cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta} = \frac{1-2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}.$$

10. Let AB represent the flagstaff, BC the tower, and let D be the position of the observer.

Then $\angle BDC = 68°\,11'$, $\angle ADB = 2°\,10'$; $\therefore$ $\angle ADC = 70°\,21'$.

Let $BC = x$ ft., then $x + 24 = DC \tan 70°\,21'$, and $DC = x \cot 68°\,11'$.

$\therefore x + 24 = x \cot 68°\,11' \tan 70°\,21'$; $\therefore$ $x + 24 = x \times 2{\cdot}8 \times {\cdot}4 = 1{\cdot}12\,x$; whence $x = 200$; that is, the height of the tower is 200 ft.

EXAMPLES. X. a. PAGE 87.

1. $\cos 135° = \cos(180° - 45°) = -\cos 45° = -\dfrac{1}{\sqrt{2}}$.

2. $\sin 150° = \sin(180° - 30°) = \sin 30° = \dfrac{1}{2}$.

3. $\tan 240° = \tan(180° + 60°) = \tan 60° = \sqrt{3}$.

4. $\operatorname{cosec} 225° = \operatorname{cosec}(180° + 45°) = -\operatorname{cosec} 45° = -\sqrt{2}$.

5. $\sin(-120°) = -\sin 120° = -\sin(180° - 60°) = -\sin 60° = -\dfrac{\sqrt{3}}{2}$.

6. $\cot(-135°) = -\cot 135° = -\cot(180° - 45°) = \cot 45° = 1$.

7. $\cot 315° = \cot(180° + 135°) = \cot 135° = -1$.

8. $\cos(-240°) = \cos 240° = \cos(180° + 60°) = -\cos 60° = -\dfrac{1}{2}$.

9. $\sec(-300°) = \sec 300° = \sec(180° + 120°) = -\sec 120° = -\sec(180° - 60°)$
$= \sec 60° = 2.$

10. $\tan\frac{3\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = -1.$

11. $\sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$

12. $\sec\frac{2\pi}{3} = \sec\left(\pi - \frac{\pi}{3}\right) = -\sec\frac{\pi}{3} = -2.$

13. $\operatorname{cosec}\left(-\frac{\pi}{6}\right) = -\operatorname{cosec}\frac{\pi}{6} = -2.$

14. $\cos\left(-\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$

15. $\cot\left(-\frac{5\pi}{6}\right) = -\cot\frac{5\pi}{6} = -\cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{\pi}{6} = \sqrt{3}.$

16. $\cos(270° + A) = \cos(180° + \overline{90° + A}) = -\cos(90° + A) = \sin A.$

17. $\cot(270° - A) = \cot(180° + \overline{90° - A}) = \cot(90° - A) = \tan A.$

18. $\sin(A - 90°) = -\sin(90° - A) = -\cos A.$

19. $\sec(A - 180°) = \sec(180° - A) = -\sec A.$

20. $\sin(270° - A) = \sin(180° + \overline{90° - A}) = -\sin(90° - A) = -\cos A.$

21. $\cot(A - 90°) = -\cot(90° - A) = -\tan A.$

22. $\sin\left(\theta - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos\theta.$

23. $\tan(\theta - \pi) = -\tan(\pi - \theta) = \tan\theta.$

24. $\sec\left(\frac{3\pi}{2} - \theta\right) = \sec\left(\pi + \overline{\frac{\pi}{2} - \theta}\right) = -\sec\left(\frac{\pi}{2} - \theta\right) = -\operatorname{cosec}\theta.$

25. Expression $= \tan A \cos A \operatorname{cosec} A = 1.$

26. Expression $= -\sin A + \sin A - (-\sin A) - (-\sin A) = 2\sin A.$

27. Expression $= \sec^2 A - \tan^2 A = 1.$

EXAMPLES. X. b. PAGE 91.

1. $\cos 480° = \cos(360° + 120°) = \cos 120° = -\frac{1}{2}.$

2. $\sin 960° = \sin(3 \times 360° - 120°) = -\sin 120° = -\frac{\sqrt{3}}{2}.$

3. $\cos 780° = \cos(2 \times 360° + 60°) = \cos 60° = \frac{1}{2}.$

4. $\sin(-870°) = \sin(-2\times360° - 150°) = -\sin 150° = -\frac{1}{2}$.

5. $\sec 900° = \sec(2\times360° + 180°) = \sec 180° = -1$.

6. $\tan(-855°) = -\tan 855° = -\tan(2\times360° + 135°)$
$= -\tan 135° = -\tan(180° - 45°) = \tan 45° = 1$.

7. $\operatorname{cosec}(-660°) = -\operatorname{cosec} 660° = -\operatorname{cosec}(2\times360° - 60°) = \operatorname{cosec} 60° = \frac{2}{\sqrt{3}}$.

8. $\cot 840° = \cot(2\times360° + 120°) = \cot 120°$
$= \cot(180° - 60°) = -\cot 60° = -\frac{1}{\sqrt{3}}$.

9. $\operatorname{cosec}(-765°) = -\operatorname{cosec} 765°$
$= -\operatorname{cosec}(2\times360° + 45°) = -\operatorname{cosec} 45° = -\sqrt{2}$.

10. $\cos 4005° = \cos(11\times360° + 45°) = \cos 45° = \frac{1}{\sqrt{2}}$.

11. $\cot 990° = \cot(3\times360° - 90°) = -\cot 90° = 0$.

12. $\sin 3015° = \sin(8\times360° + 135°) = \sin 135° = \sin(180° - 45°)$
$= \sin 45° = \frac{1}{\sqrt{2}}$.

13. $\sec 2745° = \sec(8\times360° - 135°) = \sec 135° = -\sec 45° = -\sqrt{2}$.

14. $\cos 2400° = \cos(7\times360° - 120°) = \cos 120° = -\cos 60° = -\frac{1}{2}$.

15. $\sec(-5895°) = \sec 5895° = \sec(16\times360° + 135°)$
$= \sec 135° = -\sec 45° = -\sqrt{2}$.

16. $\sin\frac{15\pi}{4} = \sin\left(4\pi - \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.

17. $\cot\frac{23\pi}{4} = \cot\left(6\pi - \frac{\pi}{4}\right) = -\cot\frac{\pi}{4} = -1$.

18. $\sec\frac{7\pi}{3} = \sec\left(2\pi + \frac{\pi}{3}\right) = \sec\frac{\pi}{3} = 2$.

19. $\cot\frac{16\pi}{3} = \cot\left(6\pi - \frac{2\pi}{3}\right) = -\cot\frac{2\pi}{3} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$.

20. $\sec\left(\frac{3\pi}{2} + \frac{\pi}{3}\right) = \sec\left(2\pi + \frac{\pi}{3} - \frac{\pi}{2}\right) = \sec\left(2\pi - \frac{\pi}{6}\right) = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$.

21. $\cos\theta = \frac{\sqrt{3}}{2} = \cos 30°$; $\therefore\ \theta = 30°$ satisfies the equation.

And $\cos 30° = \cos(360° - 30°) = \cos 330°$.

There are no angles whose boundary lines are in the second and th

quadrants which satisfy the equation since the cosine in those quadrants is negative.

$\therefore$ the positive angles are 30°, 330°.

And the negative angles are $-(360°-30°)$, $-(360°-330°)$.

That is, the angles are $\pm 30°$, $\pm 330°$.

22. $\sin\theta = -\frac{1}{2} = \sin(180°+30°) = \sin 210°$; $\therefore\ \theta = 210°$ is a solution.

Also $\sin(360°-30°) = -\sin 30° = -\frac{1}{2}$; $\therefore\ \theta = 330°$ is another solution.

Thus 210°, 330° are the positive angles.

The negative angles are $-(360°-210°)$, $-(360°-330°)$;

$\therefore$ the required angles are 210°, 330°, $-150°$, $-30°$.

23. $\tan\theta = -\sqrt{3} = \tan(180°-60°) = \tan 120°$; $\therefore\ \theta = 120°$ is a solution.

Also $\tan(360°-60°) = -\tan 60° = -\sqrt{3}$; $\therefore\ \theta = 300°$ is another solution.

Thus 120°, 300° are the positive angles.

The negative angles are $-(360°-120°)$, $-(360°-300°)$;

$\therefore$ the required angles are 120°, 300°, $-240°$, $-60°$.

24. $\cot\theta = -1 = -\cot 45° = \cot 135°$.

Also $\cot 135° = \cot(180°+135°) = \cot 315°$.

$\therefore$ the positive angles which satisfy the equation are 135°, 315°.

The negative angles are $-(360°-135°)$, $-(360°-315°)$.

$\therefore$ the required angles are 135°, 315°, $-45°$, $-225°$.

25. Let the radius vector OP start from the position OX and revolve in the positive direction till it reaches the position OP, such that $\angle POX = A$. Then let it revolve in the negative direction through an angle of 180°, reaching the position OP'.

Then POP' is a straight line, and $\angle XOP' = A - 180°$.

Draw PM, $P'M'$ perpendicular to XX'. Then the $\triangle^s$ OPM, $OP'M'$ are geometrically equal.

Then $\sec(A-180°) = \frac{OP'}{OM'} = -\frac{OP}{OM} = -\sec A$.

6. Proceed as in Art. 97. Let the radius vector first revolve from OX ough the angle A to the position OP. Again, let it revolve from OX ough 270° and then further through an angle A to the position OP'; aw PM, $P'M'$ perpendiculars to XX'. Then from the equal $\triangle^s$ OPM, $OP'M'$, e have $P'M' = -OM$, $O'M' = PM$;

$$\therefore\ \tan(270°+A) = \frac{P'M'}{O'M'} = -\frac{OM}{PM} = -\cot A.$$

27. Let OP be determined as before, and then let the radius vector turn back in the negative direction through an angle 90° to the position OP'. Draw perpendiculars as before.

Then $\cos(A-90°)=\dfrac{OM'}{OP'}=\dfrac{PM}{OP}=\sin A.$

28. First side $=\tan A-\tan A-\tan A=-\tan A=\tan(360°-A).$

29. First side $=\dfrac{\sin A}{\tan A}\cdot\dfrac{\tan A}{-\cot A}\cdot\dfrac{\cos A}{-\sin A}=\sin A.$

30. Expression $=\dfrac{-\sin A}{-\sin A}-\dfrac{-\cot A}{\cot A}+\dfrac{\cos A}{\cos A}=1+1+1=3.$

31. Expression $=\dfrac{\operatorname{cosec} A}{-\sec A}\cdot\dfrac{\cos A}{-\sin A}=\dfrac{\cos^2 A}{\sin^2 A}=\cot^2 A.$

32. Expression $=\dfrac{-\sin A\,.\sec A\,.(-\tan A)}{\sec A\,(-\sin A)\tan A}=-1.$

33. First side $=\sin\left(\pi-\dfrac{\pi}{2}+\theta\right)\sin\left(\dfrac{\pi}{2}+\pi-\theta\right)\cot\left(\pi+\dfrac{\pi}{2}+\theta\right)$

$=\sin\left(\dfrac{\pi}{2}-\theta\right)\sin\left(\dfrac{3\pi}{2}-\theta\right)\cot\left(\dfrac{\pi}{2}+\theta\right).$

34. $\sin\alpha=\sin\dfrac{11\pi}{4}=\sin\left(2\pi+\dfrac{3\pi}{4}\right)=\sin\dfrac{3\pi}{4}=\sin\dfrac{\pi}{4}=\dfrac{1}{\sqrt{2}};$

$\cos\alpha=\cos\dfrac{3\pi}{4}=-\cos\dfrac{\pi}{4}=-\dfrac{1}{\sqrt{2}}.$ Also $\tan\alpha=-1.$

$\therefore$ Expression $=\dfrac{1}{2}-\dfrac{1}{2}-2-2=-4.$

EXAMPLES. XI. a. PAGE 97.

1. $\sin(A+45°)=\sin A\cos 45°+\cos A\sin 45°=\dfrac{1}{\sqrt{2}}(\sin A+\cos A).$

2. $\cos(A+45°)=\cos A\cos 45°-\sin A\sin 45°=\dfrac{1}{\sqrt{2}}(\cos A-\sin A).$

3. $2\sin(30°-A)=2(\sin 30°\cos A-\cos 30°\sin A)=\cos A-\sqrt{3}\sin A.$

4. $\cos A=\dfrac{4}{5};\quad\therefore\ \sin A=\dfrac{3}{5}.\qquad\cos B=\dfrac{3}{5};\quad\therefore\ \sin B=\dfrac{4}{5}.$

$\therefore\ \sin(A+B)=\sin A\cos B+\cos A\sin B=1;$

$\cos(A-B)=\cos A\cos B+\sin A\sin B=\dfrac{24}{25}.$

6. $\sec A = \frac{17}{8}$; $\therefore \cos A = \frac{8}{17}$, $\sin A = \frac{15}{17}$.

$\operatorname{cosec} B = \frac{5}{4}$; $\therefore \sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$.

$$\therefore \sec(A+B) = \frac{1}{\cos A \cos B - \sin A \sin B} = -\frac{85}{36}.$$

7. $\sin 75° = \sin(90° - 15°) = \cos 15° = \cos(45° - 30°)$

$$= \cos 45° \cos 30° + \sin 45° \sin 30° = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

8. $\sin 15° = \cos(90° - 15°) = \cos 75° = \cos(45° + 30°)$

$$= \cos 45° \cos 30° - \sin 45° \sin 30° = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

9. $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta} = \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \tan\alpha + \tan\beta.$

10. $\dfrac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta} = \cot\beta - \cot\alpha.$

11. $\dfrac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta} = \dfrac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\cos\alpha\sin\beta} = \cot\beta + \tan\alpha.$

12. $\cos(A+B)\cos(A-B)$

$$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$$
$$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$
$$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$$
$$= \cos^2 A - \sin^2 B.$$

13. $\sin(A+B)\sin(A-B)$

$$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$$
$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$
$$= (1-\cos^2 A)\cos^2 B - \cos^2 A(1-\cos^2 B)$$
$$= \cos^2 B - \cos^2 A.$$

14. $\cos(45° - A) - \sin(45° + A) = \frac{1}{\sqrt{2}}\{\cos A + \sin A - \sin A - \cos A\} = 0.$

15. $\cos(45° + A) + \sin(A - 45°) = \frac{1}{\sqrt{2}}(\cos A - \sin A + \sin A - \cos A) = 0.$

16. First side $= \cos A\cos B + \sin A\sin B - \sin A\cos B - \cos A\sin B$

$$= (\cos A - \sin A)\cos B - (\cos A - \sin A)\sin B$$
$$= (\cos A - \sin A)(\cos B - \sin B).$$

17. First side $= \cos A \cos B - \sin A \sin B + \sin A \cos B - \cos A \sin B$

$= (\cos A + \sin A)\cos B - (\cos A + \sin A)\sin B$

$= (\cos A + \sin A)(\cos B - \sin B)$.

18. First side

$= 2(\sin A \cos 45° + \cos A \sin 45°)(\sin A \cos 45° - \cos A \sin 45°)$

$= 2 \times \frac{1}{\sqrt{2}}(\sin A + \cos A) \times \frac{1}{\sqrt{2}}(\sin A - \cos A)$

$= \sin^2 A - \cos^2 A$.

19. First side $= 2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos \alpha - \sin \alpha) \times \frac{1}{\sqrt{2}}(\cos \alpha + \sin \alpha)$

$= \cos^2 \alpha - \sin^2 \alpha$.

20. First side $= 2 \cdot \frac{1}{\sqrt{2}} \cdot (\cos \alpha + \sin \alpha) \times \frac{1}{\sqrt{2}}(\cos \beta - \sin \beta)$

$= \cos \alpha \cos \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta - \sin \alpha \sin \beta$

$= \cos(\alpha + \beta) + \sin(\alpha - \beta)$.

21. As in Ex. 10, it is easily shewn that the first term of the expression $= \tan \beta - \tan \gamma$.

Thus the first side

$= \tan \beta - \tan \gamma + \tan \gamma - \tan \alpha + \tan \alpha - \tan \beta = 0$.

EXAMPLES. XI. b. PAGE 100.

1. We have $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$;

$$\therefore \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1.$$

3. We have $\cot A = \frac{5}{7}$, $\cot B = \frac{7}{5}$;

$$\therefore \cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B} = 0,$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{7}{5} - \frac{5}{7}}{1+1} = \frac{12}{35}.$$

5. $\tan(45° + A) = \frac{\tan 45° + \tan A}{1 - \tan 45° \tan A} = \frac{1 + \tan A}{1 - \tan A}$.

6. $\tan(45^\circ - A) = \dfrac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A} = \dfrac{1 - \tan A}{1 + \tan A}$.

7. $\cot\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{\cot\frac{\pi}{4}\cot\theta + 1}{\cot\theta - \cot\frac{\pi}{4}} = \dfrac{\cot\theta + 1}{\cot\theta - 1}$.

8. $\cot\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{\cot\frac{\pi}{4}\cot\theta - 1}{\cot\theta + \cot\frac{\pi}{4}} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$.

9. $\tan 15^\circ = \tan(60^\circ - 45^\circ) = \dfrac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ} = \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}} = 2 - \sqrt{3}$.

10. $\cot 15^\circ = \cot(45^\circ - 30^\circ) = \dfrac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}$.

11. $\cos(A + B + C) = \cos A \cos(B + C) - \sin A \sin(B + C)$
$= \cos A \cos B \cos C - \cos A \sin B \sin C$
$- \sin A \sin B \cos C - \sin A \cos B \sin C$.

$\sin(A - B + C) = \sin(A - B)\cos C + \cos(A - B)\sin C$
$= \sin A \cos B \cos C - \cos A \sin B \cos C$
$+ \cos A \cos B \sin C + \sin A \sin B \sin C$.

12. $\tan(A - B - C) = \dfrac{\tan(A - B) - \tan C}{1 + \tan(A - B)\tan C}$

$= \dfrac{\dfrac{\tan A - \tan B}{1 + \tan A \tan B} - \tan C}{1 + \dfrac{(\tan A - \tan B)\tan C}{1 + \tan A \tan B}}$

$= \dfrac{\tan A - \tan B - \tan C - \tan A \tan B \tan C}{1 + \tan A \tan B - \tan B \tan C + \tan C \tan A}$.

13. $\cot(A + B + C) = \dfrac{\cot(A + B)\cot C - 1}{\cot C + \cot(A + B)}$

$= \dfrac{\dfrac{(\cot A \cot B - 1)\cot C}{\cot B + \cot A} - 1}{\cot C + \dfrac{\cot A \cot B - 1}{\cot B + \cot A}}$

$= \dfrac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}$.

EXAMPLES. XI. c. PAGE 101.

1. First side $=\cos(\overline{A+B}-B)=\cos A$.

2. First side $=\sin(3A-A)=\sin 2A$.

3. First side $=\cos(2\alpha-\alpha)=\cos\alpha$.

4. First side $=\cos(30°+A+\overline{30°-A})=\cos 60°=\frac{1}{2}$.

5. First side $=\sin(60°-A+\overline{30°+A})=\sin 90°=1$.

6. First side $=\cos 2\alpha\cos\alpha-\sin 2\alpha\sin\alpha=\cos(2\alpha+\alpha)=\cos 3\alpha$.

7. First side $=\tan(\overline{\alpha-\beta}+\beta)=\tan\alpha$.

8. First side $=\cot(\overline{\alpha+\beta}-\alpha)=\cot\beta$.

9. First side $=\tan(4A-3A)=\tan A$.

10. $$\cot\theta-\cot 2\theta=\frac{\cos\theta}{\sin\theta}-\frac{\cos 2\theta}{\sin 2\theta}=\frac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}$$
$$=\frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta}=\frac{\sin\theta}{\sin\theta\sin 2\theta}=\operatorname{cosec}2\theta.$$

11. $$1+\tan 2\theta\tan\theta=1+\frac{\sin 2\theta\sin\theta}{\cos 2\theta\cos\theta}=\frac{\cos\theta\sin 2\theta+\sin\theta\sin 2\theta}{\cos\theta\cos 2\theta}$$
$$=\frac{\cos(2\theta-\theta)}{\cos\theta\cos 2\theta}=\sec 2\theta.$$

12. $$1+\cot 2\theta\cot\theta=\frac{\sin 2\theta\sin\theta+\cos 2\theta\cos\theta}{\sin\theta\sin 2\theta}$$
$$=\frac{\cos(2\theta-\theta)}{\sin\theta\sin 2\theta}=\operatorname{cosec}2\theta\cot\theta.$$

13. First side $=\sin(2\theta+\theta)=\sin 3\theta$
$=\sin(4\theta-\theta)$
$=\sin 4\theta\cos\theta-\cos 4\theta\sin\theta$.

14. First side $=\cos(4\alpha+\alpha)=\cos 5\alpha$
$=\cos(3\alpha+2\alpha)$
$=\cos 3\alpha\cos 2\alpha-\sin 3\alpha\sin 2\alpha$.

EXAMPLES. XI. d. PAGE 104.

1. Here $\cos 2A=2\cos^2 A-1=-\frac{7}{9}$.

3. We have $\sin A = \frac{3}{5}$, and $\cos A = \frac{4}{5}$;

$$\therefore \sin 2A = 2 \sin A \cos A = \frac{24}{25}.$$

5. By Art. 124, $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{7}{25}$,

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{24}{25}.$$

7. See Example, Art. 122.

8. $\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{2\cos^2 A} = \tan A.$

9. $\frac{\sin 2A}{1-\cos 2A} = \frac{2\sin A\cos A}{2\sin^2 A} = \cot A.$

10. $\frac{1-\cos A}{\sin A} = \frac{2\sin^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \tan\frac{A}{2}.$

11. $\frac{1+\cos A}{\sin A} = \frac{2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \cot\frac{A}{2}.$

12. $2\operatorname{cosec} 2a = \frac{2}{\sin 2a} = \frac{1}{\sin a\cos a} = \sec a \operatorname{cosec} a.$

13. $\tan a + \cot a = \frac{\sin^2 a + \cos^2 a}{\sin a\cos a} = \frac{1}{\sin a\cos a} = 2\operatorname{cosec} 2a.$

14. $\cos^4 a - \sin^4 a = (\cos^2 a + \sin^2 a)(\cos^2 a - \sin^2 a) = \cos 2a.$

15. $\cot a - \tan a = \frac{\cos^2 a - \sin^2 a}{\sin a\cos a} = \frac{2\cos 2a}{2\sin a\cos a} = 2\cot 2a.$

16. By Art. 116, $\cot 2A = \frac{\cot A\cot A - 1}{\cot A + \cot A} = \frac{\cot^2 A - 1}{2\cot A}.$

17. $\frac{\cot A - \tan A}{\cot A + \tan A} = \frac{\frac{1}{\tan A} - \tan A}{\frac{1}{\tan A} + \tan A} = \frac{1-\tan^2 A}{1+\tan^2 A} = \cos 2A.$

18. $\frac{1+\cot^2 A}{2\cot A} = \frac{\sin^2 A + \cos^2 A}{2\cot A\sin^2 A} = \frac{1}{2\sin A\cos A} = \operatorname{cosec} 2A.$

19. $\dfrac{\cot^2 A+1}{\cot^2 A-1}=\dfrac{1+\tan^2 A}{1-\tan^2 A}=\sec 2A.$

20. $\dfrac{1+\sec\theta}{\sec\theta}=\dfrac{1}{\sec\theta}+1=\cos\theta+1=2\cos^2\dfrac{\theta}{2}.$

21. $\dfrac{\sec\theta-1}{\sec\theta}=1-\cos\theta=2\sin^2\dfrac{\theta}{2}.$

22. $\dfrac{2-\sec^2\theta}{\sec^2\theta}=\dfrac{2}{\sec^2\theta}-1=2\cos^2\theta-1=\cos 2\theta.$

23. $\dfrac{\operatorname{cosec}^2\theta-2}{\operatorname{cosec}^2\theta}=1-2\sin^2\theta=\cos 2\theta.$

24. First side $=\sin^2\dfrac{A}{2}+\cos^2\dfrac{A}{2}+2\sin\dfrac{A}{2}\cos\dfrac{A}{2}=1+\sin A.$

25. First side $=\sin^2\dfrac{A}{2}+\cos^2\dfrac{A}{2}-2\sin\dfrac{A}{2}\cos\dfrac{A}{2}=1-\sin A.$

26. $\dfrac{\cos 2\alpha}{1+\sin 2\alpha}=\dfrac{\cos^2\alpha-\sin^2\alpha}{(\sin\alpha+\cos\alpha)^2}=\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}=\dfrac{1-\tan\alpha}{1+\tan\alpha}=\tan(45^\circ-\alpha).$

27. $\dfrac{\cos 2\alpha}{1-\sin 2\alpha}=\dfrac{\cos^2\alpha-\sin^2\alpha}{(\cos\alpha-\sin\alpha)^2}=\dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}=\cot(45^\circ-\alpha).$

28. $\sin 8A=2\sin 4A\cos 4A=4\sin 2A\cos 2A\cos 4A$
$=8\sin A\cos A\cos 2A\cos 4A.$

29. $\cos 4A=2\cos^2 2A-1$
$=2(2\cos^2 A-1)^2-1$
$=8\cos^4 A-8\cos^2 A+1.$

30. Second side $=\cos 2\left(45^\circ-\dfrac{A}{2}\right)=\cos(90^\circ-A)=\sin A.$

31. $\cos^2\left(\dfrac{\pi}{4}-\alpha\right)-\sin^2\left(\dfrac{\pi}{4}-\alpha\right)=\cos 2\left(\dfrac{\pi}{4}-\alpha\right)=\cos\left(\dfrac{\pi}{2}-2\alpha\right)=\sin 2\alpha.$

32. First side $=\dfrac{1+\tan A}{1-\tan A}-\dfrac{1-\tan A}{1+\tan A}=\dfrac{4\tan A}{1-\tan^2 A}=2\tan 2A.$

33. First side $=\dfrac{1+\tan A}{1-\tan A}+\dfrac{1-\tan A}{1+\tan A}=\dfrac{2(1+\tan^2 A)}{1-\tan^2 A}$
$=2\sec 2A.$ [Art. 124.]

EXAMPLES. XI. e. PAGE 106.

4. $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = \dfrac{3\sin A - 4\sin^3 A}{\sin A} - \dfrac{4\cos^3 A - 3\cos A}{\cos A}$

$= 3 - 4\sin^2 A - 4\cos^2 A + 3$

$= 6 - 4\,(\sin^2 A + \cos^2 A) = 2.$

5. $\cot 3A = \cot(2A + A) = \dfrac{\cot 2A \,.\, \cot A - 1}{\cot 2A + \cot A}$

$= \dfrac{\dfrac{\cot^2 A - 1}{2\cot A} \,.\, \cot A - 1}{\dfrac{\cot^2 A - 1}{2\cot A} + \cot A} = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}.$

6. First side $= \dfrac{3\cos\alpha + 4\cos^3\alpha - 3\cos\alpha}{3\sin\alpha - 3\sin\alpha + 4\sin^3\alpha} = \cot^3\alpha.$

7. First side $= \dfrac{3\sin\alpha - 3\sin^3\alpha}{3\cos\alpha - 3\cos^3\alpha} = \dfrac{\sin\alpha\,(1 - \sin^2\alpha)}{\cos\alpha\,(1 - \cos^2\alpha)} = \cot\alpha.$

8. First side $= \dfrac{3\cos\alpha - 3\cos^3\alpha}{\cos\alpha} + \dfrac{3\sin\alpha - 3\sin^3\alpha}{\sin\alpha}$

$= 3 - 3\cos^2\alpha + 3 - 3\sin^2\alpha$

$= 6 - 3\,(\cos^2\alpha + \sin^2\alpha) = 3.$

9. $\sin 18° + \sin 30° = \dfrac{\sqrt{5} - 1}{4} + \dfrac{1}{2} = \dfrac{\sqrt{5} + 1}{4} = \sin 54°.$

10. $\cos 36° - \sin 18° = \dfrac{\sqrt{5} + 1}{4} - \dfrac{\sqrt{5} - 1}{4} = \dfrac{1}{2}.$

11. $\cos^2 36° + \sin^2 18° = \left(\dfrac{\sqrt{5} + 1}{4}\right)^2 + \left(\dfrac{\sqrt{5} - 1}{4}\right)^2 = \dfrac{3 + \sqrt{5}}{8} + \dfrac{3 - \sqrt{5}}{8} = \dfrac{3}{4}.$

12. $4\sin 18°\cos 36° = 4 \,.\, \dfrac{\sqrt{5} - 1}{4} \,.\, \dfrac{\sqrt{5} + 1}{4} = \dfrac{5 - 1}{4} = 1.$

EXAMPLES. XI. f. PAGE 108.

1. First side $= \dfrac{\sin 2A}{\cos 2A} - \dfrac{\sin A}{\cos A} = \dfrac{\sin 2A\cos A - \cos 2A\sin A}{\cos A\cos 2A}$

$= \dfrac{\sin(2A - A)}{\cos A\cos 2A} = \tan A\sec 2A.$

2. First side $= \dfrac{\sin 2A}{\cos 2A} + \dfrac{\cos A}{\sin A} = \dfrac{\sin 2A\sin A + \cos 2A\cos A}{\cos 2A\sin A}$

$= \dfrac{\cos A}{\cos 2A\sin A} = \cot A\sec 2A.$

3. First side $= \dfrac{2\sin^2\theta + 2\sin\theta\cos\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} = \dfrac{2\sin\theta\,(\sin\theta+\cos\theta)}{2\cos\theta\,(\sin\theta+\cos\theta)} = \tan\theta.$

4. First side $= \dfrac{2\cos^2\frac{\theta}{2} + \cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}} = \dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}.$

5. $\cos^6\alpha - \sin^6\alpha = (\cos^2\alpha - \sin^2\alpha)(\cos^4\alpha + \cos^2\alpha\sin^2\alpha + \sin^4\alpha)$

$= \cos 2\alpha\,[(\cos^2\alpha + \sin^2\alpha)^2 - \sin^2\alpha\cos^2\alpha]$

$= \cos 2\alpha\left(1 - \frac{1}{4}\sin^2 2\alpha\right).$

6. $4(\cos^6\theta + \sin^6\theta) = 4(\cos^2\theta + \sin^2\theta)(\cos^4\theta + \sin^4\theta - \sin^2\theta\cos^2\theta)$

$= 4\{(\cos^2\theta + \sin^2\theta)^2 - 3\sin^2\theta\cos^2\theta\}$

$= 4 - 3\sin^2 2\theta = 1 + 3(1 - \sin^2 2\theta)$

$= 1 + 3\cos^2 2\theta.$

7. First side $= \dfrac{4\cos^3\alpha - 3\cos\alpha + 3\sin\alpha - 4\sin^3\alpha}{\cos\alpha - \sin\alpha}$

$= \dfrac{4(\cos^3\alpha - \sin^3\alpha) - 3(\cos\alpha - \sin\alpha)}{\cos\alpha - \sin\alpha}$

$= 4(\cos^2\alpha + \sin\alpha\cos\alpha + \sin^2\alpha) - 3$

$= 4 + 2\sin 2\alpha - 3 = 1 + 2\sin 2\alpha.$

8. First side $= \dfrac{4\cos^3\alpha - 3\cos\alpha - 3\sin\alpha + 4\sin^3\alpha}{\cos\alpha + \sin\alpha}$

$= 4(\cos^2\alpha - \cos\alpha\sin\alpha + \sin^2\alpha) - 3$

$= 4 - 2\sin 2\alpha - 3 = 1 - 2\sin 2\alpha.$

9. $\dfrac{\cos\alpha + \sin\alpha}{\cos\alpha - \sin\alpha} = \dfrac{(\cos\alpha + \sin\alpha)^2}{\cos^2\alpha - \sin^2\alpha} = \dfrac{\sin 2\alpha + 1}{\cos 2\alpha} = \tan 2\alpha + \sec 2\alpha.$

10. $\dfrac{\cot\alpha - 1}{\cot\alpha + 1} = \dfrac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \dfrac{1 - \sin 2\alpha}{\cos 2\alpha}.$ [See Example 2, p. 107.]

11. $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} = \dfrac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} = \dfrac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}.$

12. $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2} = \dfrac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} = \dfrac{\cot\frac{\theta}{2} + 1}{\cot\frac{\theta}{2} - 1}.$

13. $\sec A - \tan A = \dfrac{1-\sin A}{\cos A} = \dfrac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}} = \tan\left(45° - \frac{A}{2}\right)$.

14. $\tan A + \sec A = \dfrac{\sin A + 1}{\cos A} = \dfrac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}} = \cot\left(45° - \frac{A}{2}\right)$.

15. Second side $= \dfrac{2\sin^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} = \dfrac{1-\cos\left(\frac{\pi}{2}+\theta\right)}{1+\cos\left(\frac{\pi}{2}+\theta\right)} = \dfrac{1+\sin\theta}{1-\sin\theta}$.

16. $(2\cos A+1)(2\cos A-1) = 4\cos^2 A - 1 = 2\cos 2A + 1$.

17. First side $= \dfrac{2\sin A\cos A}{2\cos^2 A} \times \dfrac{\cos A}{2\cos^2\frac{A}{2}} = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}} = \tan\dfrac{A}{2}$.

18. First side $= \dfrac{2\sin A\cos A}{2\sin^2 A} \times \dfrac{2\sin^2\frac{A}{2}}{\cos A} = \dfrac{2\sin^2\frac{A}{2}}{\sin A} = \tan\dfrac{A}{2}$.

19. First side $= \cos 3\alpha\,(3\sin\alpha - \sin 3\alpha) + \sin 3\alpha\,(3\cos\alpha + \cos 3\alpha)$
$= 3\,(\cos 3\alpha\sin\alpha + \sin 3\alpha\cos\alpha) = 3\sin 4\alpha$.

20. First side $= \dfrac{1}{4}(3\cos\alpha + \cos 3\alpha)\cos 3\alpha + \dfrac{1}{4}(3\sin\alpha - \sin 3\alpha)\sin 3\alpha$

$= \dfrac{3}{4}(\cos\alpha\cos 3\alpha + \sin\alpha\sin 3\alpha) + \dfrac{1}{4}(\cos^2 3\alpha - \sin^2 3\alpha)$

$= \dfrac{1}{4}\{3\cos(3\alpha-\alpha) + \cos 6\alpha\} = \dfrac{1}{4}(3\cos 2\alpha + \cos 6\alpha) = \cos^3 2\alpha$.

21. First side $= 3\cos 20° + \cos 60° + 3\cos 40° + \cos 120°$
$= 3\,(\cos 20° + \cos 40°)$.

22. First side $= 3\cos 10° + \cos 30° + 3\sin 20° - \sin 60°$
$= 3\,(\cos 10° + \sin 20°)$.

23. $\tan 3A = \tan(2A + A) = \dfrac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$.

Multiply up and transpose; then we obtain

$$\tan 3A - \tan 2A - \tan A = \tan 3A\tan 2A\tan A$$

24. $$\frac{\cot\theta}{\cot\theta-\cot 3\theta}+\frac{\tan\theta}{\tan\theta-\tan 3\theta}=\frac{\dfrac{1}{\tan\theta}}{\dfrac{1}{\tan\theta}-\dfrac{1}{\tan 3\theta}}+\frac{\tan\theta}{\tan\theta-\tan 3\theta}$$
$$=\frac{\tan 3\theta}{\tan 3\theta-\tan\theta}+\frac{\tan\theta}{\tan\theta-\tan 3\theta}=\frac{\tan 3\theta-\tan\theta}{\tan 3\theta-\tan\theta}=1.$$

25. $$\frac{1}{\tan 3\theta+\tan\theta}-\frac{1}{\cot 3\theta+\cot\theta}=\frac{1}{\tan 3\theta+\tan\theta}-\frac{\tan 3\theta\tan\theta}{\tan 3\theta+\tan\theta}$$
$$=\frac{1-\tan 3\theta\tan\theta}{\tan 3\theta+\tan\theta}=\frac{1}{\tan 4\theta}=\cot 4\theta.$$

EXAMPLES. XII. a. PAGE 112.

For Examples 1—16, see Examples on page 111.

17. $2\cos 2\beta\cos(\alpha-\beta)=\cos(2\beta+\alpha-\beta)+\cos(2\beta-\alpha+\beta)$
$=\cos(\beta+\alpha)+\cos(3\beta-\alpha).$

18. $2\sin 3\alpha\sin(\alpha+\beta)=\cos(3\alpha-\alpha-\beta)-\cos(3\alpha+\alpha+\beta)$
$=\cos(2\alpha-\beta)-\cos(4\alpha+\beta).$

19. $2\sin(2\theta+\phi)\cos(\theta-2\phi)=\sin(2\theta+\phi+\theta-2\phi)+\sin(2\theta+\phi-\theta+2\phi)$
$=\sin(3\theta-\phi)+\sin(\theta+3\phi).$

20. $2\cos(3\theta+\phi)\sin(\theta-2\phi)=\sin(3\theta+\phi+\theta-2\phi)-\sin(3\theta+\phi-\theta+2\phi)$
$=\sin(4\theta-\phi)-\sin(2\theta+3\phi).$

21. $\cos(60°+\alpha)\sin(60°-\alpha)=\frac{1}{2}\{\sin 120°-\sin 2\alpha\}=\frac{1}{2}\left(\frac{\sqrt{3}}{2}-\sin 2\alpha\right).$

EXAMPLES. XII. b. PAGE 114.

For Examples 1—12, see Examples on page 113.

13. $$\frac{\cos\alpha-\cos 3\alpha}{\sin 3\alpha-\sin\alpha}=\frac{2\sin 2\alpha\sin\alpha}{2\cos 2\alpha\sin\alpha}=\tan 2\alpha.$$

14. $$\frac{\sin 2\alpha+\sin 3\alpha}{\cos 2\alpha-\cos 3\alpha}=\frac{2\sin\frac{5\alpha}{2}\cos\frac{\alpha}{2}}{2\sin\frac{5\alpha}{2}\sin\frac{\alpha}{2}}=\cot\frac{\alpha}{2}.$$

15. $$\frac{\cos 4\theta-\cos\theta}{\sin\theta-\sin 4\theta}=\frac{-2\sin\frac{5\theta}{2}\sin\frac{3\theta}{2}}{-2\cos\frac{5\theta}{2}\sin\frac{3\theta}{2}}=\tan\frac{5\theta}{2}.$$

16. $\dfrac{\cos 2\theta - \cos 12\theta}{\sin 12\theta + \sin 2\theta} = \dfrac{2\sin 7\theta \sin 5\theta}{2\sin 7\theta \cos 5\theta} = \tan 5\theta.$

17. First side $= 2\cos 60° \sin A = 2 \times \dfrac{1}{2} \sin A = \sin A.$

18. First side $= 2\cos 30° \cos A = \sqrt{3} \cos A.$

19. First side $= 2\sin \dfrac{\pi}{4} \sin(-\alpha) = -2 \times \dfrac{1}{\sqrt{2}} \sin\alpha = -\sqrt{2}\sin\alpha.$

20. First side $= \dfrac{2\cos\alpha\cos(\alpha - 3\beta)}{2\sin\alpha\cos(\alpha - 3\beta)} = \cot\alpha.$

21. First side $= \dfrac{2\sin(2\theta - \phi)\sin(\theta + 2\phi)}{2\sin(2\theta - \phi)\cos(\theta + 2\phi)} = \tan(\theta + 2\phi).$

22. First side $= \dfrac{2\cos\dfrac{\alpha + 5\beta}{2}\sin\dfrac{\alpha - 3\beta}{2}}{2\cos\dfrac{\alpha + 5\beta}{2}\cos\dfrac{\alpha - 3\beta}{2}} = \tan\dfrac{\alpha - 3\beta}{2}.$

EXAMPLES. XII. c. PAGE 116.

1. $\cos 3A + \sin 2A - \sin 4A = \cos 3A - 2\cos 3A \sin A$
$= \cos 3A\,(1 - 2\sin A).$

2. $\sin 3\theta - \sin\theta - \sin 5\theta = \sin 3\theta - 2\sin 3\theta \cos 2\theta$
$= \sin 3\theta\,(1 - 2\cos 2\theta).$

3. $\cos\theta + \cos 2\theta + \cos 5\theta = \cos 2\theta + 2\cos 3\theta \cos 2\theta$
$= \cos 2\theta\,(1 + 2\cos 3\theta).$

4. $\sin\alpha - \sin 2\alpha + \sin 3\alpha = \sin\alpha + 2\cos\dfrac{5\alpha}{2}\sin\dfrac{\alpha}{2}$

$= 2\sin\dfrac{\alpha}{2}\left(\cos\dfrac{\alpha}{2} + \cos\dfrac{5\alpha}{2}\right)$

$= 4\sin\dfrac{\alpha}{2}\cos\alpha\cos\dfrac{3\alpha}{2}.$

5. $\sin 3\alpha + \sin 7\alpha + \sin 10\alpha = 2\sin 5\alpha\cos 2\alpha + \sin 10\alpha$
$= 2\sin 5\alpha\,(\cos 2\alpha + \cos 5\alpha)$
$= 2\sin 5\alpha\cos\dfrac{7\alpha}{2}\cos\dfrac{3\alpha}{2}.$

6. $\sin A + 2\sin 3A + \sin 5A = 2\sin 3A + 2\sin 3A\cos 2A$
$= 2\sin 3A\,(1 + \cos 2A)$
$= 4\sin 3A\cos^2 A.$

7. First side $= \dfrac{\sin 2\alpha + 2\cos 3\alpha \sin 2\alpha}{\cos 2\alpha + 2\cos 3\alpha \cos 2\alpha} = \dfrac{\sin 2\alpha\,(1 + 2\cos 3\alpha)}{\cos 2\alpha\,(1 + 2\cos 3\alpha)} = \tan 2\alpha.$

8. First side $= \dfrac{(\sin\alpha + \sin 5\alpha) + (\sin 2\alpha + \sin 4\alpha)}{(\cos\alpha + \cos 5\alpha) + (\cos 2\alpha + \cos 4\alpha)}$

$$= \frac{2\sin 3\alpha \cos 2\alpha + 2\sin 3\alpha \cos\alpha}{2\cos 3\alpha \cos 2\alpha + 2\cos 3\alpha \cos\alpha} = \frac{\sin 3\alpha}{\cos 3\alpha} = \tan 3\alpha.$$

9. First side $= \dfrac{2\cos 5\theta \cos 2\theta - 2\cos 3\theta \cos 2\theta}{2\cos 5\theta \sin 2\theta - 2\cos 3\theta \sin 2\theta} = \cot 2\theta.$

10. First side $= \frac{1}{2}(\sin 5A - \sin A) - \frac{1}{2}(\sin 5A - \sin 3A)$

$$= \frac{1}{2}(\sin 3A - \sin A) = \cos 2A \sin A.$$

11. First side $= \frac{1}{2}(\cos 7A + \cos 3A) - \frac{1}{2}(\cos 7A + \cos A)$

$$= \frac{1}{2}(\cos 3A - \cos A) = -\sin 2A \sin A.$$

12. First side $= \frac{1}{2}(\sin 5\theta + \sin 3\theta) - \frac{1}{2}(\sin 5\theta + \sin\theta)$

$$= \frac{1}{2}(\sin 3\theta - \sin\theta) = \cos 2\theta \sin\theta.$$

13. $\cos 5° - \sin 25° = \cos 5° - \cos 65°$
$= 2\sin 35° \sin 30° = \sin 35°.$

14. $\sin 65° + \cos 65° = \sin 65° + \sin 25°$
$= 2\sin 45° \cos 20° = \sqrt{2}\cos 20°.$

15. First side $= 2\cos 60° \cos 20° - \cos 20°$
$= \cos 20° - \cos 20° = 0.$

16. First side $= 2\cos 48° \sin 30° + \cos(180° - 48°)$
$= \cos 48° - \cos 48° = 0.$

17. $\sin^2 5A - \sin^2 2A = \sin(5A + 2A)\sin(5A - 2A) = \sin 7A \sin 3A.$

18. $\cos 2A \cos 5A = \frac{1}{2}(\cos 7A + \cos 3A) = \frac{1}{2}\left(2\cos^2\frac{7A}{2} - 1 + 1 - 2\sin^2\frac{3A}{2}\right)$

$$= \cos^2\frac{7A}{2} - \sin^2\frac{3A}{2}.$$

19. First side $= 2\sin\alpha\cos(\beta+\gamma) + 2\sin\alpha\cos(\beta-\gamma)$
$= 2\sin\alpha\{\cos(\beta+\gamma) + \cos(\beta-\gamma)\}$
$= 4\sin\alpha\cos\beta\cos\gamma.$

20 First side $= 2\sin\gamma\sin(\alpha-\beta)+2\sin(\alpha+\beta)\sin\gamma$
$= 2\sin\gamma\{\sin(\alpha-\beta)+\sin(\alpha+\beta)\}$
$= 4\sin\alpha\cos\beta\sin\gamma.$

21. First side $= 2\sin(\alpha+\beta)\cos(\alpha-\beta)-2\sin(\alpha+\beta)\cos(\alpha+\beta+2\gamma)$
$= 2\sin(\alpha+\beta)\{\cos(\alpha-\beta)-\cos(\alpha+\beta+2\gamma)\}$
$= 4\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha).$

22. First side $= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\alpha+\beta+2\gamma}{2}\cos\frac{\alpha+\beta}{2}$

$= 2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta+2\gamma}{2}\right)$

$= 4\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}.$

23. First side $= 2\sin A\{\cos 2A-\cos 120°\}$
$= 2\sin A\left\{\cos 2A+\frac{1}{2}\right\}$
$= 2\sin A\cos 2A+\sin A$
$= \sin 3A-\sin A+\sin A = \sin 3A.$

24. First side $= 2\cos\theta\left\{\cos\frac{4\pi}{3}+\cos 2\theta\right\} = 2\cos\theta\left(-\frac{1}{2}+\cos 2\theta\right)$
$= -\cos\theta+2\cos\theta\cos 2\theta$
$= -\cos\theta+\cos 3\theta+\cos\theta$
$= \cos 3\theta.$

25. First side $= \cos\theta+2\cos\frac{2\pi}{3}\cos\theta$

$= \cos\theta-\cos\theta = 0.$

26. First side $= \frac{1}{2}\{1+\cos 2A+1+\cos 2(60°+A)+1+\cos 2(60°-A)\}$

$= \frac{1}{2}\{3+\cos 2A+2\cos 120°\cos 2A\}$

$= \frac{1}{2}\{3+\cos 2A-\cos 2A\} = \frac{3}{2}.$

27. First side $= \frac{1}{2}\{3-\cos 2A-\cos 2(120°+A)-\cos 2(120°-A)\}$

$= \frac{1}{2}\{3-\cos 2A-2\cos 240°\cos 2A\}$

$= \frac{1}{2}\{3-\cos 2A+\cos 2A\} = \frac{3}{2}.$

28. $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{2}\cos 20^\circ (\cos 120^\circ + \cos 40^\circ)$

$$= \frac{1}{2}\cos 20^\circ \left(-\frac{3}{2} + 2\cos^2 20^\circ\right)$$

$$= \frac{1}{4}(4\cos^3 20^\circ - 3\cos 20^\circ) = \frac{1}{4}\cos 60^\circ = \frac{1}{8}.$$

29. $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{1}{2}\sin 20^\circ \{\cos 40^\circ - \cos 120^\circ\}$

$$= \frac{1}{2}\sin 20^\circ \left\{\frac{3}{2} - 2\sin^2 20^\circ\right\}$$

$$= \frac{1}{4}\sin 60^\circ = \frac{\sqrt{3}}{8}.$$

EXAMPLES. XII. d. PAGE 119.

1. First side $= 2\cos(A+B)\sin(A-B) + 2\sin C\cos C$

$$= -2\cos C\sin(A-B) + 2\sin(A+B)\cos C$$

$$= 2\cos C\{\sin(A+B) - \sin(A-B)\}$$

$$= 4\cos A\sin B\cos C.$$

2. First side $= 2\cos(A+B)\sin(A-B) - 2\sin C\cos C$

$$= -2\cos C\{\sin(A-B) + \sin(A+B)\}$$

$$= -4\sin A\cos B\cos C.$$

3. First side $= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right\}$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

4. First side $= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{C}{2}$

$$= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right\}$$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$$

5. First side $= 2\sin\frac{A+B}{2}\sin\frac{B-A}{2} + 2\cos^2\frac{C}{2} - 1$

$$= 2\cos\frac{C}{2}\left\{\sin\frac{B-A}{2} + \sin\frac{B+A}{2}\right\} - 1$$

$$= 4\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} - 1.$$

6. First side $= \dfrac{4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \tan\frac{B}{2}\tan\frac{C}{2}.$

7. We have $\tan\dfrac{A+B}{2}\tan\dfrac{C}{2}=1;$

$$\therefore \frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\tan\frac{C}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=1.$$

Multiply up and transpose and we obtain the required result.

8. First side $= \dfrac{2\cos^2\frac{A}{2}+2\sin\frac{B+C}{2}\sin\frac{B-C}{2}}{2\cos^2\frac{A}{2}-2\sin\frac{B+C}{2}\sin\frac{B-C}{2}}$

$$= \frac{\sin\frac{B+C}{2}+\sin\frac{B-C}{2}}{\sin\frac{B+C}{2}-\sin\frac{B-C}{2}} = \frac{2\sin\frac{B}{2}\cos\frac{C}{2}}{2\cos\frac{B}{2}\sin\frac{C}{2}} = \tan\frac{B}{2}\cot\frac{C}{2}.$$

9. First side $= 2\cos(A+B)\cos(A-B)+2\cos^2 C+4\cos A\cos B\cos C$

$$= -2\cos C\{\cos(A-B)+\cos(A+B)\}+4\cos A\cos B\cos C$$

$$= -4\cos A\cos B\cos C+4\cos A\cos B\cos C=0.$$

10. We have $\cot(A+B)=\cot(180°-C)=-\cot C;$

$$\therefore \frac{\cot A\cot B-1}{\cot A+\cot B}=-\cot C;$$

whence by multiplying up and transposing we obtain the required result.

11. First side $= \dfrac{\sin(B+C)}{\sin B\sin C}\cdot\dfrac{\sin(C+A)}{\sin C\sin A}\cdot\dfrac{\sin(A+B)}{\sin A\sin B}$

$$= \frac{\sin A\sin B\sin C}{\sin^2 A\sin^2 B\sin^2 C} = \operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C.$$

12. First side $= \dfrac{1}{2}\{3+\cos 2A+\cos 2B+\cos 2C+4\cos A\cos B\cos C\}$

$$= \frac{1}{2}(3-1), \text{ by Example 9,}$$

$$= 1.$$

13. First side $=\frac{1}{2}(3-\cos A-\cos B-\cos C)$

$=\frac{1}{2}\left(3-1-4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$ [See Ex. 3, p. 119.]

$=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

14. First side $=\frac{1}{2}\{2+2\cos^2 2A+\cos 4B+\cos 4C\}$

$=\frac{1}{2}\{2+2\cos^2 2A+2\cos 2(B+C)\cos 2(B-C)\}$

$=1+\cos 2A\{\cos 2(B+C)+\cos 2(B-C)\}$

$=1+2\cos 2A\cos 2B\cos 2C$.

15. First side $=\frac{\cot B+\cot C}{\frac{1}{\cot B}+\frac{1}{\cot C}}+\frac{\cot C+\cot A}{\frac{1}{\cot C}+\frac{1}{\cot A}}+\frac{\cot A+\cot B}{\frac{1}{\cot A}+\frac{1}{\cot B}}$

$=\cot B\cot C+\cot C\cot A+\cot A\cot B$

$=1$. [See Ex. 10.]

16. First side $=\frac{\tan A\tan B\tan C}{16\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}$

$=\frac{\sin A\sin B\sin C}{\cos A\cos B\cos C}\cdot\frac{1}{16\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}$

$=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}\cdot 2\sin\frac{B}{2}\cos\frac{B}{2}\cdot 2\sin\frac{C}{2}\cos\frac{C}{2}}{\cos A\cos B\cos C\cdot 16\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}$

$=\frac{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{2\cos A\cos B\cos C}$.

EXAMPLES. XII. e. PAGE 121.

1. First side $=\frac{1}{2}\{\sin 2\alpha-\sin 2\beta+\sin 2\beta-\sin 2\gamma+\sin 2\gamma-\sin 2\delta+\sin 2\delta-\sin 2\alpha\}=0$.

2. First side $=\cot\gamma-\cot\beta+\cot\alpha-\cot\gamma+\cot\beta-\cot\alpha=0$.

[See

3. First side $= \dfrac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}}$

$= \dfrac{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}} = \cot\frac{\alpha}{2}\cot\frac{\beta}{2}.$

4. First side $= \sin\alpha(\cos\beta\cos\gamma - \sin\beta\sin\gamma) - \sin\beta(\cos\alpha\cos\gamma - \sin\alpha\sin\gamma)$

$= \cos\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \cos\gamma\sin(\alpha-\beta).$

5. First side $= \cos\alpha(\cos\beta\cos\gamma - \sin\beta\sin\gamma) - \cos\beta(\cos\alpha\cos\gamma - \sin\alpha\sin\gamma)$

$= \sin\gamma(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sin\gamma\sin(\alpha-\beta).$

6. First side $= \cos A\cos 2A + \sin A\sin 2A - (\sin A\cos 2A + \cos A\sin 2A)$

$= \cos(2A - A) - \sin(2A + A)$

$= \cos A - \sin 3A.$

7. $a\cos 2\theta + b\sin 2\theta = \dfrac{a(1-\tan^2\theta)}{1+\tan^2\theta} + \dfrac{2b\tan\theta}{1+\tan^2\theta},$ [Art. 124.]

$= \dfrac{a(a^2-b^2)}{a^2+b^2} + \dfrac{2ab^2}{a^2+b^2} = a\cdot\dfrac{a^2+b^2}{a^2+b^2} = a.$

8. $\sin 2A + \cos 2A = \dfrac{2\tan A}{1+\tan^2 A} + \dfrac{1-\tan^2 A}{1+\tan^2 A} = \dfrac{(1+\tan A)^2 - 2\tan^2 A}{1+\tan^2 A}.$

9. $\sin 4A = 2\sin 2A\cos 2A = \dfrac{4\tan A(1-\tan^2 A)}{(1+\tan^2 A)^2}.$

10. We have $A + B = 45°;$

$\therefore\ \tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B} = 1;$

$\therefore\ \tan A + \tan B + \tan A\tan B = 1.$

$\therefore\ (1+\tan A)(1+\tan B) = 1 + \tan A + \tan B + \tan A\tan B = 2.$

11. First side $= \dfrac{\cos(15°-A)\cos(15°+A) + \sin(15°-A)\sin(15°+A)}{\sin(15°-A)\cos(15°+A)}$

$= \dfrac{2\cos 2A}{\sin 30° - \sin 2A} = \dfrac{4\cos 2A}{1 - 2\sin 2A}.$

12. First side $= \dfrac{\cos^2(15°+A) + \sin^2(15°+A)}{\sin(15°+A)\cos(15°+A)} = \dfrac{2}{\sin(30°+2A)}$

$= \dfrac{2}{\frac{1}{2}\cos 2A + \frac{\sqrt{3}}{2}\sin 2A} = \dfrac{4}{\cos 2A + \sqrt{3}\sin 2A}.$

13. First side $= \dfrac{\sin(A+30^\circ)\sin(A-30^\circ)}{\cos(A+30^\circ)\cos(A-30^\circ)}$

$$= \frac{\cos 60^\circ - \cos 2A}{\cos 60^\circ + \cos 2A} = \frac{1-2\cos 2A}{1+2\cos 2A}.$$

14. First side $= (4\cos^2 A - 1)(2\cos 2A - 1)$

$$= (2\cos 2A + 1)(2\cos 2A - 1)$$
$$= 4\cos^2 2A - 1 = 2\cos 4A + 1.$$

15. We have $\tan(\theta+\phi+\psi) = \dfrac{\tan\theta+\tan\phi+\tan\psi-\tan\theta\tan\phi\tan\psi}{1-\tan\theta\tan\phi-\tan\phi\tan\psi-\tan\psi\tan\theta}$,

$$\therefore \text{ if } \theta+\phi+\psi = 0,$$

we have $$\tan\theta+\tan\phi+\tan\psi = \tan\theta\tan\phi\tan\psi.$$

Now putting $$\theta = \beta-\gamma,\ \phi = \gamma-\alpha,\ \psi = \alpha-\beta,$$

we have at once the required identity.

16. First side

$$= 2\sin\frac{\beta-\alpha}{2}\cos\frac{\alpha+\beta-2\gamma}{2} + \sin(\alpha-\beta) + 4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}$$

$$= -2\sin\frac{\alpha-\beta}{2}\left\{\cos\frac{\alpha+\beta-2\gamma}{2} - \cos\frac{\alpha-\beta}{2}\right\} + 4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}$$

$$= -4\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\gamma-\beta}{2} + 4\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2} = 0.$$

17. First side $= \dfrac{1}{2}\{\cos 2(\beta-\gamma) + \cos 2(\gamma-\alpha) + 2\cos^2(\alpha-\beta) + 2\}$

$$= \frac{1}{2}\{2\cos(\alpha-\beta)\cos(\beta-2\gamma+\alpha) + 2\cos^2(\alpha-\beta) + 2\}$$
$$= \cos(\alpha-\beta)[\cos(\beta-2\gamma+\alpha) + \cos(\alpha-\beta)] + 1$$
$$= 1 + 2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta).$$

18. First side $= 1 + \dfrac{1}{2}(\cos 2\alpha + \cos 2\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta)$

$$= 1 + \cos(\alpha+\beta)\cos(\alpha-\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta)$$
$$= 1 + \cos(\alpha+\beta)\{\sin\alpha\sin\beta - \cos\alpha\cos\beta\}$$
$$= 1 - \cos^2(\alpha+\beta) = \sin^2(\alpha+\beta).$$

19. First side $= 1 - \dfrac{1}{2}(\cos 2\alpha + \cos 2\beta) + 2\sin\alpha\sin\beta\cos(\alpha+\beta)$

$$= 1 - \cos(\alpha+\beta)\{\cos(\alpha-\beta) - 2\sin\alpha\sin\beta\}$$
$$= 1 - \cos^2(\alpha+\beta) = \sin^2(\alpha+\beta).$$

20. First side $= \cos 12° + 2\cos 72° \cos 12°$

$$= \cos 12° \left(1 + \frac{\sqrt{5}-1}{2}\right) = \cos 12° \left(\frac{\sqrt{5}+1}{2}\right)$$

$$= 2\cos 12° \cos 36° = \cos 24° + \cos 48°.$$

21. $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$ [Ex. 2, p. 121.]

$$= 4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}.$$

22. Second side $= 2\cos\frac{\pi-B}{4}\left[\cos\frac{2\pi+A+C}{4} + \cos\frac{A-C}{4}\right]$

$$= 2\cos\frac{A+C}{4}\cos\left(\frac{\pi}{2}+\frac{A+C}{4}\right) + 2\cos\frac{A+C}{4}\cos\frac{A-C}{4}$$

$$= -2\cos\frac{A+C}{4}\sin\frac{A+C}{4} + \cos\frac{A}{2} + \cos\frac{C}{2}$$

$$= -\sin\frac{A+C}{2} + \cos\frac{A}{2} + \cos\frac{C}{2}$$

$$= \cos\frac{A}{2} - \cos\frac{B}{2} + \cos\frac{C}{2}.$$

23. This may be done in the same way as the two preceding examples. The following solution exhibits another method.

From Example 3 of Art. 135,

if $\alpha+\beta+\gamma=\pi$, then $\cos\alpha+\cos\beta+\cos\gamma = 1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.

Put $\alpha = \frac{\pi}{2} - \frac{A}{2}$, $\beta = \frac{\pi}{2} - \frac{B}{2}$, $\gamma = \frac{\pi}{2} - \frac{C}{2}$;

then $\alpha+\beta+\gamma=\pi$, and after substituting for α, β, γ,

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$$

24. First side $= \dfrac{2\sin(\alpha+\beta)\cos(\alpha-\beta)+\sin 2\gamma}{2\sin(\alpha+\beta)\cos(\alpha-\beta)+\sin 2\gamma} = \dfrac{\cos(\alpha-\beta)+\sin\gamma}{\cos(\alpha-\beta)+\sin\gamma}$

$$= \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)} = \cot\alpha\cot\beta.$$

25. Here $\tan(\alpha+\beta) = \tan\left(\frac{\pi}{2}-\gamma\right) = \cot\gamma$;

$$\therefore \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{1}{\tan\gamma}.$$

Multiply up, and transpose.

EXAMPLES. XIII. a. PAGE 128.

1. $\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{225+49-169}{210} = \dfrac{105}{210} = \dfrac{1}{2}$; $\therefore\ C = 60°$.

2. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{9+25-49}{30} = -\dfrac{15}{30} = -\dfrac{1}{2}$; $\therefore\ A = 120°$.

3. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{5^2(3+1-1)}{5^2 2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$; $\therefore\ A = 30°$.

Also $a = c$; $\therefore\ C = A = 30°$; hence $B = 180° - A - C = 120°$.

4. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{961+98-625}{434\sqrt{2}} = \dfrac{1}{\sqrt{2}}$; $\therefore\ A = 45°$.

5. Let $a = 2$, $b = 2\frac{2}{3}$, $c = 3\frac{1}{3}$.

Then $$\cos C = \frac{4+\frac{64}{9}-\frac{100}{9}}{\frac{32}{3}} = 0;\ \therefore\ C = 90°.$$

6. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{4+6-(4+2\sqrt{3})}{4\sqrt{6}} = \dfrac{3-\sqrt{3}}{2\sqrt{6}} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\therefore\ A = 75°$;

$\cos B = \dfrac{c^2+a^2-b^2}{2ac} = \dfrac{6+4+2\sqrt{3}-4}{2\sqrt{6}(\sqrt{3}+1)} = \dfrac{3+\sqrt{3}}{\sqrt{2}(3+\sqrt{3})} = \dfrac{1}{\sqrt{2}}$; $\therefore\ B = 45°$;

$\therefore\ C = 180° - 75° - 45° = 60°$.

7. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{4+4-2\sqrt{3}-2}{4(\sqrt{3}-1)} = \dfrac{3-\sqrt{3}}{2(\sqrt{3}-1)} = \dfrac{\sqrt{3}}{2}$; $\therefore\ A = 30°$;

$\cos B = \dfrac{c^2+a^2-b^2}{2ca} = \dfrac{4-2\sqrt{3}+2-4}{2\sqrt{2}(\sqrt{3}-1)} = \dfrac{1-\sqrt{3}}{\sqrt{2}(\sqrt{3}-1)} = -\dfrac{1}{\sqrt{2}}$;

$\therefore\ B = 135°$;

$\therefore\ C = 180° - 30° - 135° = 15°$.

8. $\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{64+25-19}{80} = \dfrac{70}{80} = \dfrac{7}{8}$; $\therefore\ C = 28°\,56'$.

9. $\cos C = \dfrac{4^2+5^2-7^2}{2\times4\times5} = \dfrac{16+25-49}{40} = -\dfrac{8}{40} = -\dfrac{1}{5} = -\cos 78°\,27'$;

$\therefore\ C = 180° - 78°\,27' = 101°\,33'$.

10. $c^2 = a^2+b^2-2ab\cos C = 4+4+2\sqrt{3} - 2.2(\sqrt{3}+1)\dfrac{1}{2} = 6$.

11. $b^2=c^2+a^2-2ca\cos B=9+25-2\,.\,3\,.\,5\left(-\frac{1}{2}\right)=9+25+15=49;$

$\therefore\ b=7.$

12. $a^2=b^2+c^2-2bc\cos A=49+36-2\,.\,7\,.\,6\times\frac{1}{4}=64;$

$\therefore\ a=8.$

13. $a^2=b^2+c^2-2bc\cos A=64+121-2\,.\,8\,.\,11\left(-\frac{1}{16}\right)$

$=64+121+11=196;$

$\therefore\ a=14.$

14. $b^2=c^2+a^2-2ca\cos B=9+49-2\,.\,3\,.\,7\left(-\frac{23}{42}\right)=9+49+23=81;$

$\therefore\ b=9.$

15. $b^2=c^2+a^2-2ca\cos B=48-24\sqrt{3}+24-2\,.\,2\sqrt{6}\,.\,2\sqrt{3}\,(\sqrt{3}-1)\,.\,\frac{\sqrt{3}-1}{2\sqrt{2}}$

$=48-24\sqrt{3}+24-4\,.\,3\,.\,(4-2\sqrt{3})=24;$

$\therefore\ b=2\sqrt{6};$ whence $A=B=75°$, and $C=30°$.

16. $a^2=b^2+c^2-2bc\cos A=4+6+2\sqrt{5}-2\,.\,2\,(\sqrt{5}+1)\,\frac{\sqrt{5}-1}{4}$

$=4+6+2\sqrt{5}-4=6+2\sqrt{5};$

$\therefore\ a=\sqrt{5}+1;$ whence $C=A=72°$, and $B=36°$.

17. $C=75°;$ whence $a=c.$

Also $$a=\frac{b\sin A}{\sin B}=\frac{\sqrt{8}\,(\sqrt{3}+1)\,2}{2\sqrt{2}}=2\sqrt{3}+2.$$

18. $c=\frac{b\sin C}{\sin B}=\sqrt{6}\,.\,\frac{\sqrt{3}-1}{2\sqrt{2}}\,.\,\frac{2}{\sqrt{3}}=\sqrt{3}-1.$

Also $A=105°;$ whence $a=\frac{b\sin A}{\sin B}=\sqrt{6}\,.\,\frac{\sqrt{3}+1}{2\sqrt{2}}\,.\,\frac{2}{\sqrt{3}}=\sqrt{3}+1.$

$C=30°;$ whence $a=\frac{c\sin A}{\sin C}=\frac{\sqrt{2}\times 2}{\sqrt{2}}=2;$

$b=\frac{a\sin B}{\sin A}=2\,.\,\frac{\sqrt{3}+1}{2\sqrt{2}}\,.\,\sqrt{2}=\sqrt{3}+1.$

J. Here $\frac{c}{a}=\frac{\sin C}{\sin A}=\frac{\sin 75°}{\sin 45°}=\frac{\sqrt{3}+1}{2\sqrt{2}}\Big/\frac{1}{\sqrt{2}}=\frac{\sqrt{3}+1}{2}.$

21. $\sin A=\frac{a\sin C}{c}=\frac{2}{2\sqrt{3}}\,.\,\frac{\sqrt{3}}{2}=\frac{1}{2};$

$\therefore\ b=a\cos C+c\cos A=2\left(-\frac{1}{2}\right)+2\sqrt{3}\,.\,\frac{\sqrt{3}}{2}=3-1=2.$

22. $\sin A = \dfrac{a \sin B}{b} = \dfrac{3}{3\sqrt{3}} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{1}{2}$;

$$\therefore \quad c = a \cos B + b \cos A = 3 \cdot \frac{1}{2} + 3\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 6.$$

23. We have $(b+c)^2 - a^2 = 3bc$;

$$\therefore \quad \frac{b^2+c^2-a^2}{2bc} = \frac{1}{2}; \text{ whence } A = 60°.$$

24. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{12+6-(12+6\sqrt{3})}{2 \cdot 2\sqrt{3} \cdot \sqrt{6}}$

$$= \frac{6(1-\sqrt{3})}{12\sqrt{2}} = -\frac{\sqrt{3}-1}{2\sqrt{2}} = -\cos 75°;$$

$$\therefore A = 105°; \text{ similarly } C = 30°.$$

$$\therefore \sin B = \frac{b \sin C}{c} = \frac{2\sqrt{3}}{2\sqrt{6}} = \frac{1}{\sqrt{2}}; \text{ whence } B = 45°.$$

25. The sides are proportional to $\sqrt{3}+1$, $\sqrt{3}-1$, $\sqrt{6}$;

$$\therefore \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{4-2\sqrt{3}+6-(4+2\sqrt{3})}{2\sqrt{6}(\sqrt{3}-1)}$$

$$= \frac{6-4\sqrt{3}}{2\sqrt{6}(\sqrt{3}-1)} = \frac{2\sqrt{3}(\sqrt{3}-2)}{2\sqrt{3} \cdot \sqrt{2}(\sqrt{3}-1)} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{1-\sqrt{3}}{2\sqrt{2}} = -\cos 75°;$$

$$\therefore A = 105°.$$

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6}}{\sqrt{3}+1} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore C = 60°, \text{ and } B = 15°.$$

26. Here $b = \dfrac{\sqrt{6}+\sqrt{2}}{4}$, $c = \dfrac{\sqrt{6}-\sqrt{2}}{4}$, $A = 60°$;

$$\therefore a^2 = \frac{(\sqrt{6}+\sqrt{2})^2 + (\sqrt{6}-\sqrt{2})^2}{16} - \frac{2.4}{16} \cdot \frac{1}{2} = \frac{12}{16} = \frac{3}{4};$$

$$\therefore a = \frac{\sqrt{3}}{2}.$$

$$\sin C = \frac{c}{a} \sin A = \frac{\sqrt{6}-\sqrt{2}}{4} \cdot \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}};$$

whence $$C = 15°, \text{ and } B = 105°.$$

EXAMPLES. XIII. b. PAGE 132.

1. $\sin B = \dfrac{b \sin A}{a} = \sqrt{3} \times \dfrac{1}{2} = \dfrac{\sqrt{3}}{2}$.

$\therefore B = 60°$ or $120°$; and since $a < b$, both these values are admissible. Hence $C = 90°$ or $30°$.

$$c = \frac{a \sin C}{\sin A} = 2 \text{ or } 1, \text{ on reduction.}$$

2. $\sin B = \dfrac{b \sin C}{c} = \dfrac{3\sqrt{2}}{2\sqrt{3}} \times \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2}$.

$\therefore B = 60°$ or $120°$; and since $c < b$, both these values are admissible. Hence $A = 75°$ or $15°$.

$$a = \frac{b \sin A}{\sin B} = 3 + \sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

3. $\sin A = \dfrac{a \sin C}{c} = \dfrac{2}{\sqrt{6}} \times \dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{2}}$.

$\therefore A = 45°$, the other value being inadmissible, since $c > a$. Hence $B = 75°$.

$$b = \frac{a \sin B}{\sin A} = \sqrt{3} + 1.$$

4. $\sin C = \dfrac{c}{a} \sin A = \dfrac{5}{2} \times \dfrac{1}{2} = \dfrac{5}{4}$, which is impossible. Thus there is no triangle with the given parts.

5. $\sin C = \dfrac{c}{b} \sin B = \dfrac{2\sqrt{3}}{\sqrt{6}} \cdot \dfrac{1}{2} = \dfrac{1}{\sqrt{2}}$.

$\therefore C = 45°$ or $135°$; and since $b < c$, both values are admissible. Hence $A = 105°$ or $15°$.

$$a = \frac{b \sin A}{\sin B} = \sqrt{3}(\sqrt{3} + 1), \text{ or } \sqrt{3}(\sqrt{3} - 1), \text{ on reduction.}$$

6. $\sin C = \dfrac{c}{b} \sin B = \dfrac{3 + \sqrt{3}}{3\sqrt{2}} \cdot \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$.

$\therefore C = 75°$ or $105°$; and since $b < c$, both values are admissible. Hence $A = 45°$ or $15°$.

$$a = \frac{b \sin A}{\sin B} = 2\sqrt{3}, \text{ or } 3 - \sqrt{3}, \text{ on reduction.}$$

7. $\sin A = \dfrac{a}{c} \sin C = \dfrac{3 + \sqrt{3}}{3 - \sqrt{3}} \cdot \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$.

$\therefore A = 75°$ or $105°$; and since $c < a$, both values are admissible. Hence $B = 90°$ or $60°$.

$$b = \frac{c \sin B}{\sin C} = 2\sqrt{6}, \text{ or } 3\sqrt{2}, \text{ on reduction.}$$

8. $\sin B = \frac{b}{a}\sin A = \frac{4(\sqrt{5}+1)}{4} \cdot \frac{\sqrt{5}-1}{4} = 1.$

$\therefore B = 90°$, and there is no ambiguity.

$$c^2 = b^2 - a^2 = (b+a)(b-a) = (8+\sqrt{80})\sqrt{80}$$
$$= 4(2+\sqrt{5}) \cdot 4\sqrt{5} = 16\sqrt{5}(2+\sqrt{5})$$
$$\therefore c = 4\sqrt{5+2\sqrt{5}}.$$

9. $\sin A = \frac{a}{b}\sin B = \frac{3\sqrt{2}}{2\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}.$

$\therefore A = 60°$ or $120°$; but *neither* of these values is admissible as in each case the sum of the angles would be greater than 180°. Thus the triangle is impossible.

EXAMPLES. XIII. c. PAGE 134.

1. Follows at once from Art. 137.

2. The first side $= b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2 = a^2 + b^2 + c^2.$

3. The first side $= \frac{a^2+b^2-c^2}{2} - \frac{a^2+c^2-b^2}{2} = b^2 - c^2.$

4. The first side $= (b\cos A + a\cos B) + (c\cos B + b\cos C)$
$+ (c\cos A + a\cos C) = c + a + b.$

5. The first side $= a(1-\cos C) + c(1-\cos A)$
$= a + c - (a\cos C + c\cos A)$
$= a + c - b.$

6. The second side $= \frac{a\cos B}{a\cos C} = \frac{\cos B}{\cos C}.$

7. The second side $= \frac{a\sin C}{c\cos A} = \frac{c\sin A}{c\cos A} = \tan A.$

8. Put $k = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; then

the first side $= k(\sin B + \sin C)\sin\frac{A}{2}$

$= 2k\sin\frac{B+C}{2} \cdot \cos\frac{B-C}{2} \cdot \sin\frac{A}{2}$

$= 2k\sin\frac{A}{2}\cos\frac{A}{2} \cdot \cos\frac{B-C}{2}$

$= k\sin A \cdot \cos\frac{B-C}{2}$

$= a\cos\frac{B-C}{2}.$

9. The first side $= \dfrac{k(\sin A + \sin B)}{k \sin C} \sin^2 \dfrac{C}{2}$

$$= \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}} \sin^2 \frac{C}{2}$$

$$= \cos \frac{A-B}{2} \sin \frac{C}{2}$$

$$= \cos \frac{A-B}{2} \cos \frac{A+B}{2}$$

$$= \frac{\cos A + \cos B}{2}.$$

10. The first side $= k\{\sin A \sin (B-C) + \ldots\ldots + \ldots\ldots\}$

$= k\{\sin (B+C) \sin (B-C) + \ldots\ldots + \ldots\ldots\}$

$= k\{\sin^2 B - \sin^2 C + \ldots\ldots + \ldots\ldots\} = 0.$

11. The second side $= \dfrac{\sin^2 A - \sin^2 B}{\sin^2 C}$

$$= \frac{\sin (A+B) \sin (A-B)}{\sin (A+B) \sin (A+B)} = \frac{\sin (A-B)}{\sin (A+B)}.$$

12. The second side $= \dfrac{\sin^2 A - \sin^2 B}{\sin^2 C - \sin^2 A} = \dfrac{\sin (A+B) \sin (A-B)}{\sin (C+A) \sin (C-A)}$

$$= \frac{\sin C \sin (A-B)}{\sin B \sin (C-A)} = \frac{c \sin (A-B)}{b \sin (C-A)}.$$

EXAMPLES. XIII. d. PAGE 136.

1. Let ABC be the triangle, in which $B = C = 2A$, and $a = 2$;

then $B + C + A = 5A = 180°$; $\therefore\ A = 36°$, $B = C = 72°$;

and $b = c = \dfrac{a}{\sin A} \sin B = \dfrac{a \sin 2A}{\sin A} = 2a \cos A = 4 \cdot \dfrac{\sqrt{5}+1}{4} = \sqrt{5}+1.$

2. $A = 180° - B - C = 60°$:

If AD be the perpendicular, then $c = AD \operatorname{cosec} 45° = 3\sqrt{2}$,

$$b = AD \operatorname{cosec} 75° = \frac{6\sqrt{2}}{\sqrt{3}+1} = 3(\sqrt{6} - \sqrt{2})$$

$$a = BD + DC = AD \cot 45° + AD \cot 75° = 3 + 3(2 - \sqrt{3}) = 9 - 3\sqrt{3}.$$

3. $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{28-16\sqrt{3}+24-12\sqrt{3}-4}{4\sqrt{2}\,(9-5\sqrt{3})} = \dfrac{48-28\sqrt{3}}{4\sqrt{2}\,(9-5\sqrt{3})}$

$$= \frac{(12-7\sqrt{3})\,(9+5\sqrt{3})}{\sqrt{2}\,.\,6} = \frac{3-3\sqrt{3}}{6\sqrt{2}} = -\frac{\sqrt{3}-1}{2\sqrt{2}};$$

$$\therefore\ A = 180^\circ - 75^\circ = 105^\circ.$$

$$\sin C = \frac{c}{a}\sin A = \frac{3\sqrt{2}-\sqrt{6}}{2}\cdot\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}}{2}\cdot\frac{(\sqrt{3}-1)\,(\sqrt{3}+1)}{2} = \frac{\sqrt{3}}{2};$$

$$\therefore\ C = 60^\circ;$$

and $$B = 180^\circ - 105^\circ - 60^\circ = 15^\circ.$$

4. We have $$b-a=2,\ ab=4;$$

$$\therefore\ b+a = 2\sqrt{5},\text{ rejecting the negative sign,}$$

$$\therefore\ a = \sqrt{5}-1,\ b = \sqrt{5}+1;$$

$$\therefore\ \sin B = \frac{b\sin A}{a} = \frac{\sqrt{5}+1}{\sqrt{5}-1}\cdot\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}+1}{4};$$

$$\therefore\ B = 54^\circ \text{ or } 126^\circ;$$

$$\therefore\ C = 108^\circ \text{ or } 36^\circ.$$

5. $\sin c = \dfrac{c\sin B}{b} = 150\cdot\dfrac{1}{2}\cdot\dfrac{1}{50\sqrt{3}} = \dfrac{\sqrt{3}}{2};$

$\therefore\ C = 60^\circ$ or 120°; both values being admissible since $b<c$;

$\therefore\ A = 90^\circ$ or 30°.

In the first case, $a = \sin A\,.\,\dfrac{b}{\sin B} = 50\sqrt{3}\times 2 = 100\sqrt{3}.$

In the second case, $a = b = 50\sqrt{3}$.

If $$B = 30^\circ,\ C = 150^\circ,\ b = 75^\circ;$$

$$\sin C = \frac{150}{75}\times\sin 30^\circ = 1;$$

$$\therefore\ C = 90^\circ,\text{ and there is no ambiguity.}$$

6. We have at once from a figure, $\sin B = \dfrac{\sqrt{5}-1}{4}$; whence $B = 18^\circ$;

$$\therefore\ C = 180^\circ - 36^\circ - 18^\circ = 126^\circ.$$

7. Let ABC be the triangle, and let $B = 22\frac{1}{2}^\circ$, $C = 112\frac{1}{2}^\circ$, and let AD be the perpendicular from A on BC. Then $A = 180^\circ - 22\frac{1}{2}^\circ - 112\frac{1}{2}^\circ = 45^\circ$; also

$$AD = AB\sin 22\tfrac{1}{2}^\circ,$$

$$BC = \frac{AB}{\sin 112\frac{1}{2}^\circ}\cdot\sin 45^\circ = \frac{AB}{\sqrt{2}\cos 22\frac{1}{2}^\circ} = \frac{2AB\sin 22\frac{1}{2}^\circ}{\sqrt{2}\sin 45^\circ} = 2AD.$$

That is, the altitude is half the base.

8. $\dfrac{\sin A}{\sin B}=\dfrac{a}{b}=2$; and $\sin A=2\sin B=\sin 3B$;

$$\therefore \frac{3\sin B-4\sin^3 B}{\sin B}=2, \text{ or } 3-4\sin^2 B=2.$$

$$\therefore \sin B=\frac{1}{2}, \text{ rejecting the negative value.}$$

$$\therefore B=30°,\ A=90°,\ C=60°.$$

Also $$c=a\sin C=\frac{a\sqrt{3}}{2}.$$

9. Let C be the greatest angle, then

$$\cos C=\frac{(2x+3)^2+(x^2+2x)^2-(x^2+3x+3)^2}{2(2x+3)(x^2+2x)}$$

$$=\frac{-2x^3-7x^2-6x}{2(2x+3)(x^2+2x)}=-\frac{(2x+3)(x+2)}{2(2x+3)(x+2)}=-\frac{1}{2}.$$

Thus the greatest angle is 120°.

10. First side $=b\cos C+c\cos B-a\cos C-c\cos A=a-b.$

Now $$\frac{a-b}{c}=\frac{\sin A-\sin B}{\sin C}=\frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{C}{2}\cos\dfrac{C}{2}}$$

$$\therefore =\frac{\sin\dfrac{A-B}{2}}{\cos\dfrac{C}{2}}=\sin\frac{A-B}{2}\operatorname{cosec}\frac{A+B}{2}.$$

11. $$\frac{b+c}{a}=\frac{\sin B+\sin C}{\sin A}=\frac{\sin B+\sin(A+B)}{\sin A}=\frac{2\sin\left(B+\dfrac{A}{2}\right)\cos\dfrac{A}{2}}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}.$$

$$\therefore (b+c)\sin\frac{A}{2}=a\sin\left(\frac{A}{2}+B\right).$$

12. $$\frac{a+b}{b+c}=\frac{\sin A+\sin B}{\sin B+\sin C}=\frac{\sin B+C+\sin B}{\sin B+\sin C}=\frac{2\sin\left(B+\dfrac{C}{2}\right)\cos\dfrac{C}{2}}{2\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}}$$

$$=\frac{\sin\left(B+\dfrac{C}{2}\right)\cos\dfrac{C}{2}}{\cos\dfrac{A}{2}\cos\dfrac{B-C}{2}}.$$

13. First side $= \dfrac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)} = \dfrac{1-(\cos^2 A-\sin^2 B)}{1-(\cos^2 A-\sin^2 C)}$

$= \dfrac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C} = \dfrac{a^2+b^2}{a^2+c^2}$.

14. We have $c^4-2(a^2+b^2)c^2+a^4+a^2b^2+b^4=0$;

$\therefore\ (c^2-a^2+ab+b^2)(c^2-a^2-ab+b^2)=0$;

$\therefore\ c^2=a^2+ab+b^2$, or a^2-ab+b^2;

But $c^2=a^2+b^2+2ab\cos C$;

$\therefore\ 2\cos C=1$, or -1;

$\therefore\ C=60°$, or $120°$.

15. See figure of the Ambiguous Case on page 131.

(1) $c_1-c_2=B_1B_2=2B_1D=2a\cos B_1$.

(2) $\cos\dfrac{C_1-C_2}{2}=\cos B_1CD=\dfrac{CD}{CB_1}=\dfrac{b\sin A}{a}$.

(3) c_1, c_2 are the roots of the quadratic

$c^2-2b\cos A\,.\,c+b^2-a^2=0$; [Art. 150.]

$\therefore\ c_1+c_2=2b\cos A$; $c_1c_2=b^2-a^2$.

$\therefore\ c_1^2+c_2^2-2c_1c_2\cos 2A=c_1^2+c_2^2-2c_1c_2(2\cos^2 A-1)$

$=(c_1+c_2)^2-4(b^2-a^2)\cos^2 A$

$=4b^2\cos^2 A-4b^2\cos^2 A+4a^2\cos^2 A$

$=4a^2\cos^2 A$.

(4) We have $C_1+C_2=2\angle ACD$;

$C_1-C_2=2\angle B_1CD$;

$\therefore\ \sin\dfrac{C_1+C_2}{2}\sin\dfrac{C_1-C_2}{2}=\sin ACD\sin B_1CD$

$=\cos CAB_1\cos CB_1A$

$=\cos A\cos B$.

16. See figure of Art. 148 (iii). We have $A=45°$; $\therefore\ \angle ACD=45°$.

Hence $DC=DA=\dfrac{c_1+c_2}{2}$; also $DB_2=\dfrac{c_1-c_2}{2}$;

$\therefore\ CB_2^2=\left(\dfrac{c_1+c_2}{2}\right)^2+\left(\dfrac{c_1-c_2}{2}\right)^2=\dfrac{c_1^2+c_2^2}{2}$;

$\therefore\ \cos^2 B_2CD=\dfrac{CD^2}{CB_2^2}=\dfrac{(c_1+c_2)^2}{2(c_1^2+c_2^2)}$;

$\therefore\ \cos B_1CB_2=2\cos^2 B_2CD-1=\dfrac{(c_1+c_2)^2}{c_1^2+c_2^2}-1=\dfrac{2c_1c_2}{c_1^2+c_2^2}$.

17. From the given condition we have

$$\sin C\cos A+2\cos C\sin C=\sin B\cos A+2\cos B\sin B,$$

or
$$\cos A\,(\sin C-\sin B)=\sin 2B-\sin 2C.$$

This easily reduces to

$$\cos A\sin\frac{B-C}{2}\cos\frac{B+C}{2}=\cos A\sin\frac{B-C}{2}\cos\frac{B-C}{2}.$$

Now $\cos\dfrac{B+C}{2}$ cannot equal $\cos\dfrac{B-C}{2}$; hence we must have

$$\cos A=0, \text{ which gives } A=90°;$$

or
$$\sin\frac{B-C}{2}=0, \text{ in which case } B=C.$$

18. Since a, b, c are in A.P.; we have $a-b=b-c$;

$$\therefore\ \sin A-\sin B=\sin B-\sin C;$$

$$\therefore\ 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}=2\sin\frac{B-C}{2}\cos\frac{B+C}{2};$$

$$\therefore\ \frac{\sin\dfrac{A-B}{2}}{\sin\dfrac{A}{2}\sin\dfrac{B}{2}}=\frac{\sin\dfrac{B-C}{2}}{\sin\dfrac{B}{2}\sin\dfrac{C}{2}};$$

$$\therefore\ \cot\frac{B}{2}-\cot\frac{A}{2}=\cot\frac{C}{2}-\cot\frac{B}{2};$$

That is
$$\cot\frac{A}{2},\ \cot\frac{B}{2},\ \cot\frac{C}{2} \text{ are in A.P.}$$

19. Let $k=\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$; then

$$\text{first side}=\frac{k^2\sin A\ \sin\overline{B+C}\sin\overline{B-C}}{\sin B+\sin C}+\ldots\ldots+\ldots\ldots$$

$$=k^2\left[\frac{\sin A\,(\sin^2 B-\sin^2 C)}{\sin B+\sin C}+\ldots\ldots+\ldots\ldots\right]$$

$$=k^2[\sin A\,(\sin B-\sin C)+\ldots\ldots+\ldots\ldots]=0.$$

MISCELLANEOUS EXAMPLES. D. PAGE 138.

1. (1) $\tan 2\theta\cot\theta-1=\dfrac{2}{1-\tan^2\theta}-1=\dfrac{1+\tan^2\theta}{1-\tan^2\theta}=\sec 2\theta.$

(2) $\sin\alpha-\cot\theta\cos\alpha=\dfrac{\sin\alpha\sin\theta-\cos\alpha\cos\theta}{\sin\theta}=-\operatorname{cosec}\theta\cos(\alpha+\theta).$

2. $c^2 = a^2 + b^2 - 2ab\cos C = 48^2 + 35^2 - 48 \times 35$

$= 13^2 + 48 \times 35 = 1849;$

$\therefore\ c = 43.$

3. Here $\tan\alpha = \sqrt{\left(\frac{17}{8}\right)^2 - 1} = \frac{15}{8};$

also $\tan\beta = \sqrt{\left(\frac{17}{15}\right)^2 - 1} = \frac{8}{15} = \cot\alpha;$

$\therefore\ \alpha + \beta = 90°;$

$\therefore\ \tan(\alpha+\beta) = \infty,$ and $\operatorname{cosec}(\alpha+\beta) = 1.$

4. The expression $= \frac{2\sin 8\alpha\cos 15\alpha}{2\sin 8\alpha\cos 6\alpha} = \frac{\cos 15\alpha}{\cos 6\alpha} = -1,$

since $15\alpha = \pi - 6\alpha.$

5. First side $= \frac{1}{2}(\sin 3\theta - \sin\theta + \sin 5\theta - \sin 3\theta + \sin 7\theta - \sin 5\theta)$

$= \frac{1}{2}(\sin 7\theta - \sin\theta)$

$= \sin 3\theta\cos 4\theta.$

6. $a^2 = 2 + 4 + 2\sqrt{3} - 2(\sqrt{3}+1) = 4;$ whence $a = 2.$

Again $\sin B = \frac{b\sin A}{a} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2};$

$\therefore\ B = 30°,$ or $150°;$ but the latter value is inadmissible since c is the greatest side. Therefore $C = 180° - (45° + 30°) = 105°.$

7. (1) $2\sin^2 36° = 1 - \cos 72° = 1 - \sin 18°$

$= 1 - \frac{\sqrt{5}-1}{4} = \frac{5-\sqrt{5}}{4} = \sqrt{5}\sin 18°.$

(2) $4\sin 36°\cos 18° = 2(\sin 54° + \sin 18°)$

$= \frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} = \sqrt{5}.$

8. $\frac{\sin 3\alpha}{\sin\alpha} + \frac{\cos 3\alpha}{\cos\alpha} = \frac{\sin 3\alpha\cos\alpha + \cos 3\alpha\sin\alpha}{\sin\alpha\cos\alpha} = \frac{2\sin 4\alpha}{\sin 2\alpha} = 4\cos 2\alpha.$

9. $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{4 + 4 - 8 + 4\sqrt{3}}{8} = \frac{\sqrt{3}}{2};$

$\therefore\ A = 30°.$

$\therefore\ B = C = \frac{1}{2}(180° - 30°) = 75°.$

10. (1) Each expression easily reduces to $\dfrac{\sin a}{\sin 3a}$.

(2) $\cos a + \cos 2a + \cos 3a = 2\cos\dfrac{3a}{2}\cos\dfrac{a}{2} + 2\cos^2\dfrac{3a}{2} - 1$

$$= 2\cos\frac{3a}{2}\left(\cos\frac{a}{2} + \cos\frac{3a}{2}\right) - 1$$

$$= 4\cos a\cos\frac{a}{2}\cos\frac{3a}{2} - 1.$$

11. (1) First side $= 2b^2 \sin C\cos C + 2c^2 \sin B\cos B$

$= 2b\sin C\,(b\cos C + c\cos B)$

$= 2ab\sin C = 2bc\sin A.$

(2) First side $= k\,(\sin A\sin\overline{B-C} + \ldots\ldots + \ldots\ldots)$

$= k\,(\sin B + \overline{C}\sin\overline{B-C} + \ldots\ldots + \ldots\ldots)$

$= k\,(\sin^2 B - \sin^2 C + \ldots\ldots + \ldots\ldots) = 0.$

12. $\tan(A+B) = \tan(360° - \overline{C+D})$;

$$\therefore \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\frac{\tan C + \tan D}{1 - \tan C\tan D}.$$

$\therefore \tan A + \tan B + \tan C + \tan D = \tan C\tan D\,(\tan A + \tan B)$

$+ \tan A\tan B\,(\tan C + \tan D)$;

or $\tan A + \tan B + \tan C + \tan D$

$= \tan A\tan B\tan C\tan D\,(\cot A + \cot B + \cot C + \cot D).$

EXAMPLES. XIV. a. PAGE 145.

For Examples **1—3** see Arts. 151, 152.

For Examples **4—7** see Arts. 162, 163.

8. $\log 768 = \log(2^8 \times 3) = 8\log 2 + \log 3 = 2{\cdot}8853613.$

9. $\log 2352 = \log(2^4 \times 3 \times 7^2) = 4\log 2 + \log 3 + 2\log 7 = 3{\cdot}3714373.$

10. $\log 35{\cdot}28 = \log\left(\dfrac{2^3 \times 3^2 \times 7^2}{10^2}\right) = 3\log 2 + 2\,(\log 3 + \log 7 - 1)$

$= {\cdot}90309 + 2\,({\cdot}4771213 + {\cdot}845098 - 1)$

$= {\cdot}90309 + {\cdot}6444386 = 1{\cdot}5475286.$

11. $\log\sqrt{6804} = \dfrac{1}{2}\log(2^2 \times 3^5 \times 7) = \dfrac{1}{2}\,(2\log 2 + 5\log 3 + \log 7)$

$= \dfrac{1}{2}\,({\cdot}60206 + 2{\cdot}3856065 + {\cdot}845098)$

$= 1{\cdot}9163822.$

12. $\log \sqrt[5]{\cdot 00162} = \frac{1}{5}\log\left(\frac{2\times 3^4}{10^5}\right) = \frac{1}{5}(\log 2 + 4\log 3 - 5)$

$= \frac{1}{5}(\bar{3}\cdot 2095152) = \bar{1}\cdot 441903.$

13. $\log \cdot 021\dot{7} = \log\frac{217-21}{9000} = \log\frac{196}{9000} = \log\frac{7^2}{15^2\times 10}$

$= 2(\log 7 - \log 3 - \log 5) - 1$

$= 2(\cdot 8450980 - 1\cdot 1760913) - 1 = \bar{2}\cdot 3380134.$

14. $\log\cos 60° = \log\left(\frac{1}{2}\right) = -\log 2 = -\cdot 30103 = \bar{1}\cdot 69897.$

15. $\log \sin^3 60° = 3\log\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}\log 3 - 3\log 2$

$= \cdot 7156819 - \cdot 90309 = \bar{1}\cdot 8125919.$

16. $\log \sqrt[3]{\sec 45°} = \frac{1}{3}\log\sqrt{2} = \frac{1}{6}\log 2 = \cdot 0501716.$

17. The expression $= \log\left[\left(\frac{15}{8}\right)^2 \times \frac{162}{25} \times \left(\frac{4}{9}\right)^3\right]$

$= \log\left(\frac{15\times 15\times 81\times 2\times 64}{64\times 25\times 81\times 9}\right) = \log 2.$

18. The expression

$= 16(1 - 2\log 3) - 4(2\log 5 - 3\log 2 - \log 3) - 7(3\log 2 + 1 - 4\log 3)$

$= 9 - 9\log 2 - 8\log 5 = 1 - \log 2 = \cdot 69897.$

19. $\log x = \frac{1}{7}\log 7 = \cdot 1207283. \quad \therefore\ x = 1\cdot 320469.$

20. $\log x = \frac{1}{3}\log\left(\frac{2^2\times 3^2\times 7^2}{10^3}\right) = \frac{1}{3}(2\log 2 + 2\log 3 + 2\log 7 - 8)$

$= \frac{1}{3}(3\cdot 2464986 - 8) = \bar{2}\cdot 4154995.$

$\therefore\ x = \cdot 0260315.$

21. $\log x = \cdot 5527899 + \bar{2}\cdot 5527899 + 1\cdot 1842633 = \cdot 2898431.$

22. $\log x = \frac{1}{3}\log\frac{11}{10^5} + 2\log\frac{11^2}{10^2} + \frac{4}{3}\log\frac{11^3}{10^2} - \log(11^2\times 10^5)$

$= \frac{1}{3}(\log 11 - 5) + 4(\log 11 - 1) + \frac{4}{3}(3\log 11 - 2) - 2\log 11 - 5.$

$= \frac{19}{3}\log 11 - \frac{40}{3} = \frac{1}{3}(\bar{21}\cdot 7864613) = \bar{7}\cdot 2621538.$

23. Since $\left(\frac{21}{20}\right)^{300} = \left(\frac{3\times 7}{2^2\times 5}\right)^{300}$;

$$\therefore \log\left(\frac{21}{20}\right)^{300} = 300(\log 3+\log 7-1-\log 2)$$
$$= 396{\cdot}66579 - 390{\cdot}30900 = 6{\cdot}35679;$$

$\therefore \left(\frac{21}{20}\right)^{300}$ has 7 digits in its integral part.

Since $\left(\frac{126}{125}\right)^{1000} = \left(\frac{2\times 3^2\times 7}{5^3}\right)^{1000}$;

$$\therefore \log\left(\frac{126}{125}\right)^{1000} = 1000(\log 2+2\log 3+\log 7-3+3\log 2)$$
$$= 1000(4\log 2+2\log 3+\log 7-3)$$
$$= 3{\cdot}4606;$$

$\therefore \left(\frac{126}{125}\right)^{1000}$ has 4 digits in its integral part.

24. 7^4 is the smallest number whose logarithm has characteristic 4.

7^3 is the smallest number whose logarithm has characteristic 3.

$\therefore$ the required number is $7^4 - 7^3 = 2058$.

EXAMPLES. XIV. b. PAGE 149.

1. $$\log x = \frac{2}{3}\log\left(\frac{147\times 375}{126\times 16}\right) = \frac{2}{3}\log\left(\frac{7\times 5^3}{2^5}\right)$$
$$= \frac{2}{3}\log\frac{7\times 1000}{16\times 16} = \frac{2}{3}(3+\log 7-8\log 2)$$
$$= \frac{2}{3}(3{\cdot}8450980 - 2{\cdot}4082400) = \frac{2}{3}(1{\cdot}4368580) = {\cdot}9579053;$$
$$\therefore x = 9{\cdot}076226.$$

2. $$\log x = \frac{1}{3}\log(3^3\times 2\times 7) + \frac{1}{2}\log(3^3\times 2^2)$$
$$- \frac{1}{6}\log(3^2\times 7\times 2^4) - \frac{1}{3}\log(2\times 3^3)$$
$$= \frac{1}{2}\log 3 + \frac{1}{3}\log 2 + \frac{1}{6}\log 7 = {\cdot}4797536;$$
$$\therefore x = 3{\cdot}01824.$$

·1003433
·2385606
·1408497
·4797536

3. $\log x = \frac{1}{2}\log(10\times 2^2\times 3^3)+\frac{5}{3}\log(2^3\times 3\div 100)+\log(10\times 3^4)$

$=\frac{43}{6}\log 3+6\log 2-\frac{11}{6}=\frac{1}{6}(20{\cdot}5162159-11)+1{\cdot}8061800$

$=1{\cdot}5860360+1{\cdot}8061800=3{\cdot}3922160;$

$\therefore\ x=2467{\cdot}266.$

4. Let x be the value; then $20^x=800.$

$\therefore\ x\log 20=\log 800;$

$\therefore\ x=\frac{\log 800}{\log 20}=\frac{2+3\log 2}{1+\log 2}=2{\cdot}23.$

5. Here $3^x=49$, or $x\log 3=2\log 7.$

$\therefore\ x=\frac{2\log 7}{\log 3}=3{\cdot}54.$

6. Here $125^x=4000$, or $x\log 5^3=\log(2^2\times 10^3).$

$\therefore\ x=\frac{3+2\log 2}{3\log 5}=1{\cdot}72.$

7. $\log x=\frac{40}{3}\log\left(\frac{378}{10^5}\right)=\frac{40}{3}(\log 2+3\log 3+\log 7-5)=\bar{33}{\cdot}699892.$

$\therefore$ the number of ciphers is 32.

$\log x=50\log\frac{259}{9990}=50\log\frac{7}{270}$

$=50(\log 7-3\log 3-1)=50(\cdot 8450980-2{\cdot}4313639)$

$=50(\bar{2}{\cdot}4137341)=\overline{80}{\cdot}6867050.$

$\therefore$ the number of ciphers is 79.

8. Let x be the base; then $x^3=11000;$

$\therefore\ 3\log x=\log 11+3=4{\cdot}0413927;$

$\therefore\ \log x=1{\cdot}3471309.$

But $\log 222398=5{\cdot}3471309;$

$\therefore\ x=22{\cdot}2398.$

9. Here $(x-1)\log 2=\log 5=1-\log 2.$

$\therefore\ x\log 2=1;$

$\therefore\ x=\frac{1}{\log 2}=\frac{1}{\cdot 30103}=3{\cdot}32.$

10. Here $(x-4)\log 3=\log 7.$

$\therefore\ x-4=\frac{\log 7}{\log 3}=1{\cdot}77;$

$\therefore\ x=5{\cdot}77.$

11. Here $(1-x)\log 5=(x-3)(\log 2+\log 3)$;

$\therefore\ x(\log 2+\log 3+\log 5)=3\log 2+3\log 3+\log 5$;

$$\therefore\ x=\frac{3\cdot0334239}{1\cdot4771213}=2\cdot05.$$

12. Put $\log 2=a$, $\log 5=b$; then

$$bx=-ay,\ b(2+y)=a(2-x).$$

From these equations we obtain

$$x=\frac{2a}{a+b}=\frac{2\log 2}{\log 2+\log 5}=\frac{2\log 2}{\log 10}=2\log 2=\cdot60206,$$

$$y=-\frac{ax}{b}=-2\log 5=-1\cdot39794.$$

13. Put $\log 2=a$, $\log 3=b$; then

$$ax=by,\ a(y+1)=b(x-1).$$

From these equations we obtain

$$x=\frac{b}{b-a}=\frac{\log 3}{\log 3-\log 2}=2\cdot71,$$

$$y=\frac{a}{b-a}=\frac{\log 2}{\log 3-\log 2}=x-1=1\cdot71.$$

14. We have $2\log 2+\log 7=a$, $\log 3+\log 7=b$, $2-2\log 2=c$.

$\therefore\ 2+\log 7=a+c$, so that $\log 3=b-a-c+2$.

$\therefore\ \log 27=3(b-a-c+2)$.

Again $\log 224=\log(2^5\times 7)=5\log 2+\log 7$.

$$=\frac{5}{2}(2-c)+(a+c-2)=\frac{1}{2}(2a-3c+6).$$

15. We have $\log(2\times 11^2)=a$, $\log(2^3\times 10)=b$, $\log(3^2\times 5)=c$.

$\therefore\ 2\log 11+\log 2=a$, $3\log 2+1=b$, $2\log 3+1-\log 2=c$.

From the last two equations

$$b+c-2=2\log 3+2\log 2=\log 36.$$

Again, $\log 66=\log 6+\log 11=\frac{1}{2}(b+c-2)+\frac{a-\log 2}{2}$

$$=\frac{1}{2}\left[b+c-2+a-\frac{1}{3}(b-1)\right]=\frac{1}{6}(3a+2b+3c-5).$$

MISCELLANEOUS EXAMPLES. E. PAGE 150.

1. First side $=\frac{1}{2}(\cos 60^\circ+\cos 2A-\cos 120^\circ-\cos 2A)$

$$=\frac{1}{2}(\cos 60^\circ-\cos 120^\circ)=\frac{1}{2}.$$

2. See Ex. 1, p. 118, and Ex. 3, p. 119.

3. $b^2=a^2+c^2-2ac\cos B=4+2-2.2.\sqrt{2}.\frac{\sqrt{3}+1}{2\sqrt{2}}=4-2\sqrt{3}=(\sqrt{3}-1)^2.$

$$\therefore\ b=\sqrt{3}-1.$$

$$\sin A=\frac{a}{b}\sin B=\frac{2}{\sqrt{3}-1}\cdot\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{1}{\sqrt{2}};$$

$\therefore A=45°$, or $135°$; but a is the greatest side, so that $A=135°$, and $C=30°$.

4. Each side easily reduces to 1.

5. Here $\frac{a}{b}=\frac{\cos A}{\cos B}$; but in any triangle $\frac{a}{b}=\frac{\sin A}{\sin B}$.

$$\therefore\ \frac{\sin A}{\sin B}=\frac{\cos A}{\cos B},\ \text{or}\ \sin(A-B)=0;$$

$$\therefore\ A=B,\ \text{and the triangle is isosceles.}$$

6. (1) $\text{First side}=\frac{1}{2}(\cos 2\theta-\cos 4\theta+\cos 4\theta-\cos 6\theta+\cos 6\theta$

$$-\cos 8\theta+\cos 8\theta-\cos 10\theta)$$

$$=\frac{1}{2}(\cos 2\theta-\cos 10\theta)=\sin 6\theta\sin 4\theta.$$

(2) $\text{First side}=\frac{2\sin 2\alpha\cos\alpha+2\sin 6\alpha\cos\alpha}{2\cos 2\alpha\cos\alpha+2\cos 6\alpha\cos\alpha}$

$$=\frac{\sin 2\alpha+\sin 6\alpha}{\cos 2\alpha+\cos 6\alpha}=\frac{2\sin 4\alpha\cos 2\alpha}{2\cos 4\alpha\cos 2\alpha}=\tan 4\alpha.$$

7. $\text{First side}=\frac{\cos 3\alpha\cos\alpha+\sin 3\alpha\sin\alpha}{\sin\alpha\cos\alpha}=\frac{2\cos(3\alpha-\alpha)}{2\sin\alpha\cos\alpha}$

$$=\frac{2\cos 2\alpha}{\sin 2\alpha}=2\cot 2\alpha.$$

8. $c^2=a^2+b^2-2ab\cos C=a^2+(4-2\sqrt{3})a^2-a^2(\sqrt{3}-1)\sqrt{3}=(2-\sqrt{3})a^2.$

$$\therefore\ a^2=(2+\sqrt{3})c^2,\ \text{and}\ \sin^2 A=(2+\sqrt{3})\sin^2 C;$$

$$\therefore\ \sin^2 A=\frac{2+\sqrt{3}}{4}=\frac{4+2\sqrt{3}}{8};$$

$$\therefore\ \sin A=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Hence $A=75°$, or $105°$, and the latter value must be taken, as A would make the triangle isosceles. Hence also $B=45°$.

9. $\tan 4\alpha=\frac{2\tan 2\alpha}{1-\tan^2 2\alpha}$; substitute $\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$.

10. (1) First side $= a^2(1-2\sin^2 B)+b^2(1-2\sin^2 A)$
$= a^2+b^2-2a^2\sin^2 B-2b^2\sin^2 A = a^2+b^2-4a^2\sin^2 B$
$= a^2+b^2-4a\sin B\,.\,b\sin A = a^2+b^2-4ab\sin A\sin B.$

(2) First side $= 2bc(1+\cos A)+\ldots\ldots+\ldots\ldots$
$= (2bc+b^2+c^2-a^2)+\ldots\ldots+\ldots\ldots = (a+b+c)^2.$

11. From the equation $c^4-2c^2(a^2+b^2)+(a^4+b^4)=0$ we have

$$\{c^2-(a^2+ab\sqrt{2}+b^2)\}\{c^2-(a^2-ab\sqrt{2}+b^2)\}=0.$$

Equating the two factors separately to zero, we get

$$\frac{a^2+b^2-c^2}{2ab} = \frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}};$$

whence $C = 45°$ or $135°$.

12. We have $2\cos\frac{3(A+B)}{2}\cos\frac{3(A-B)}{2}+\cos 3C = 1;$

$$\therefore\ 2\cos\left(270°-\frac{3C}{2}\right)\cos\frac{3(A-B)}{2}+\cos 3C = 1;$$

$$\therefore\ -2\sin\frac{3C}{2}\cos\frac{3(A-B)}{2}+1-2\sin^2\frac{3C}{2} = 1;$$

or
$$-2\sin\frac{3C}{2}\left[\cos\frac{3(A-B)}{2}+\sin\frac{3C}{2}\right]=0;$$

$$2\sin\frac{3C}{2}\left[\cos\frac{3(A-B)}{2}-\cos\frac{3(A+B)}{2}\right]=0;$$

$$4\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}=0.$$

Since A, B, C are the angles of a triangle we must have one of the angles $\frac{3A}{2}$, $\frac{3B}{2}$, or $\frac{3C}{2}$ equal to 180°. That is, one of the angles of the triangle must be 120°.

EXAMPLES. XV. a. PAGE 155.

1.

log 49517	=	4·6947543
log 49516	=	4·6947456
diff. for 1	=	·87
		·34
		3 48
		26 1
diff. for ·34	=	29 58
log 49516	=	4·694756
log 49516·34	=	4·694786

$\therefore$ log 4951634 = 6·694786.

2.

log 3·4714	=	·5405047
log 3·4713	=	·5404921
diff. for ·0001	=	126
		·026
		756
		2 52
diff. for ·0000026	=	3 276
log 3·4713	=	·5404921
log 3·4713026	=	·5404924

3. $\log 28497 = 4{\cdot}4547991$
$\log 28496 = 4{\cdot}4547839$
diff. for 1 = 152
$\cdot 14$
6 | 08
15 | 2
diff. for $\cdot 14$ = 21 | 28
$\log 28496 = 4{\cdot}4547839$
$\log 28496{\cdot}14 = 4{\cdot}4547860$
$\therefore \log 2849614 = 6{\cdot}4547860.$

4. $\log 57{\cdot}634 = 1{\cdot}7606788$
$\log 57{\cdot}633 = 1{\cdot}7606712$
diff. for $\cdot 001$ = 76
$\cdot 25$
3 | 80
15 | 2
diff. for $\cdot 00025$ = 19 | 00
$\log 57{\cdot}633 = 1{\cdot}7606712$
$\log 57{\cdot}63325 = 1{\cdot}7606731$

5. $\log 60814 = 4{\cdot}7840036$ | 6 | 43 | 2
5 | 3 | 60
$\therefore \log 6081465 = 6{\cdot}7840083$

6. $\log x = 4{\cdot}7461735$
$\log 55740 = 4{\cdot}7461670$
diff. = 65,
and diff. for 1 = 78;
$\therefore$ prop^{l}. increase $= \frac{65}{78} = \frac{5}{6} = {\cdot}83;$
$\therefore x = 55740{\cdot}83.$

7. $\log x = 2{\cdot}8283676$
$\log 67354 = 2{\cdot}8283634$
diff. = 42,
and diff. for 1 = 64;
$\therefore$ prop^{l}. increase $= \frac{42}{64} = \frac{21}{32} = {\cdot}66;$
$\therefore x = 673{\cdot}5466.$

8. $\log x = \bar{2}{\cdot}0288435$
$\log {\cdot}010686 = \bar{2}{\cdot}0288152$
diff. = 283,
and diff. for 1 = 406;
$\therefore$ prop^{l}. increase $= \frac{283}{406} = {\cdot}7;$
$\therefore x = {\cdot}0106867.$

9. $\log x = \bar{3}{\cdot}9184377$
$\log {\cdot}0082877 = \bar{3}{\cdot}9184340$
diff. = 37,
and diff. for 1 = 52;
$\therefore$ prop^{l}. increase $= \frac{37}{52} = {\cdot}71;$
$\therefore x = {\cdot}008287771.$

10. $\log x = \bar{1}{\cdot}4034508$
$\log {\cdot}25319 = \bar{1}{\cdot}4034465$
diff. = 43,
$\therefore$ prop^{l}. increase $= \frac{43}{172} = \frac{1}{4} = {\cdot}25;$
$\therefore x = {\cdot}2531925.$

11. $\log x = \frac{1}{7}\log 142{\cdot}71$
$= \frac{1}{7}(2{\cdot}1544544)$
$= {\cdot}3077792$
$\log 2{\cdot}0313 = {\cdot}3077741$
diff. = 51,
and diff. for 1 = 213;
$\therefore$ prop^{l}. increase $= \frac{51}{213} = {\cdot}24;$
$\therefore x = 2{\cdot}031324.$

12. $\log 13{\cdot}894 = 1{\cdot}1428273$ | 9 | 281 | 7
2 | 6 | 26
8 | $\bar{1}{\cdot}1428561$
$\log x = {\cdot}1428570$
$\log 1{\cdot}3894 = {\cdot}1428273$
diff. = 297
and diff. for 1 = 313;
$\therefore$ prop^{l}. increase $= \frac{297}{313} = {\cdot}95;$
$\therefore x = 1{\cdot}389495.$

13. $\log 24244 = 5{\cdot}3846043$

$7 \quad 125|3$

$14|5{\cdot}3846168$

$\log x = {\cdot}3846155$

$\log 2{\cdot}4244 = {\cdot}3846043$

$\text{diff.} = \quad 112,$

$\therefore$ prop^{l}. increase $= \frac{112}{179} = {\cdot}63;$

$\therefore\ x = 2{\cdot}424463.$

14. $\log 20691 = 6{\cdot}3157815$

$3 \quad 63|0$

$8 \quad 16|80$

$20|6{\cdot}3157895$

$\log x = {\cdot}3157895$

$\log 2{\cdot}0691 = {\cdot}3157815$

$\text{diff.} = \quad 80,$

$\therefore$ prop^{l}. increase $= \frac{80}{210} = \frac{8}{21} = {\cdot}38;$

$\therefore\ x = 2{\cdot}069138.$

EXAMPLES. XV. b. PAGE 159.

1. $\sin 38° 3' = {\cdot}6163489;$ diff. for $60'' = 2291.$

prop^{l}. increase $= \frac{35}{60} \times 2291 = \quad 1336$

$\quad\quad\quad

10. $L\sin\theta - L\sin 44°\,17' = 176$;
diff. for $60'' = 1295$;
and $\dfrac{176}{1295}\times 60'' = 8''$;
$\therefore\ \theta = 44°\,17'\,8''$.

11. $L\cos 55°\,30' - L\cos\theta = 1205$;
diff. for $60'' = 1838$;
and $\dfrac{1205}{1838}\times 60'' = 39''$;
$\therefore\ \theta = 55°\,30'\,39''$.

12. $L\tan 24°\,50' = 9{\cdot}6653662$; diff. for $60'' = 3313$.

prop^{l}. increase $= \dfrac{52{\cdot}5}{60}\times 3313 =$ 2899

$9{\cdot}6656561$

13. The required angle is $42{\cdot}5''$ *less* than $40°\,5'$;

$\therefore$ prop^{l}. *increase* $= \dfrac{42{\cdot}5}{60}\times 1502 =$ 1064

$L\operatorname{cosec} 40°\,5' = 10{\cdot}1911808$

$L\operatorname{cosec} 40°\,4'\,17{\cdot}5'' = \overline{10{\cdot}1912872}$

EXAMPLES. XV. c. PAGE 161.

1.

$\log 300{\cdot}26$	$= 2{\cdot}4774975$		
1	15		
8	11	6	
$\log {\cdot}0078915$	$= \bar{3}{\cdot}8971596$		
1	6		
9	5	0	
4		22	
	${\cdot}3746609$		
$\log 2{\cdot}3695$	$= {\cdot}3746567$		
	42		
2	37		

Thus the product $= 2{\cdot}36952$.

2.

$\log 235{\cdot}67$	$= 2{\cdot}3723043$		
8	148		
3	5	6	
$\log 357{\cdot}84$	$= 2{\cdot}5536889$		
3	36		
8	9	8	
	$4{\cdot}9260131$		
$\log 84336$	$= 4{\cdot}9260130$		

Thus the product is 84336.

3.

log 153·24 = 2·1853721
1 28
9 25 | 6
log 2·8632 = ·4568517
5 76
0 0
3 | 46
log ·075836 = $\bar{2}$·8798754
4 23
6 3 | 4
1·5221148
log 33·274 = 1·5221050
98
7 91
70
5 65

Thus the product is 33·27475.

4.

log 1·0304 = ·0130059
0 0
5 21 | 1
1 | 42
·0130081
subtract 1·4328656
$\bar{2}$·5801425
log ·038031 = $\bar{2}$·5801377
48
4 46
20
2 23

log 27·093 = 1·4328571
5 81
2 3 | 2
4 | 64
1·4328656

Thus the quotient is ·03803142.

5.

log 357·83 = 2·5536767
6 73
4 4 | 8
2·5536845
$\bar{3}$·5037539
5·0499306
log 11218 = 4·0499154
152
4 155

log ·0031897 = $\bar{3}$·5037498
3 41
$\bar{3}$·5037539

Thus the quotient is 112184.

6. $\log 21{\cdot}856 = 1{\cdot}3395707$ | $\log {\cdot}017834 = \bar{2}{\cdot}2512488$.

3 | 60 — 5 | 122

2 | 40

$1{\cdot}3395771$ — $\bar{2}{\cdot}2512610$

subtract $\bar{2}{\cdot}2512610$

$\log x = 3{\cdot}0883161$

$\log 1225{\cdot}5 = 3{\cdot}0883133$

0 | 280

8 | 284

$\therefore\ x = 1225{\cdot}508.$

7.

$\log 3{\cdot}7895 = {\cdot}5785819$

6 | 69

$\log {\cdot}053687 = \bar{2}{\cdot}7298691$

2 | 16

$\bar{1}{\cdot}3084595$

$\log {\cdot}0072916 = \bar{3}{\cdot}8628228$

$1{\cdot}4456367$

$\log 27{\cdot}902 = 1{\cdot}4456353$

0 | 140

9 | 140

Thus the required value is 27·90209.

8. $\log {\cdot}83410 = \bar{1}{\cdot}9212181$

0 | 0

3 | 1 | 6

9 | 47

$\bar{1}{\cdot}9212183$

3

$\bar{1}{\cdot}7636549$

$\log {\cdot}58030 = \bar{1}{\cdot}7636526$

23

3 | 22

Thus the cube is ·580303.

9. $\log 15063 = 4{\cdot}1779115$

0 | 0

1 | 2 | 9

8 | 2 | 30

5 | $4{\cdot}1779120$

$\cdot 8355824$

$\log 6{\cdot}8482 = {\cdot}8355764$

60

9 | 58

20

3 | 19

Thus the fifth root is 6·848293.

10.

log 384·73	= 2·5851561	log 15·732	= 1·1967839
1	11	4	111
5 | 2·5851572		13 | 1·1967950	
·5170314		·0920612	
log 3·2887	= ·5170243	log 1·2361	·0920536
	71		76
5	66	2	70
	50		60
4	53	2	70

Thus $\sqrt[5]{384{\cdot}73} = 3{\cdot}288754$. Thus $\sqrt[13]{15{\cdot}7324} = 1{\cdot}236122$.

11.

log 1034·3	= 3·0146465 |	log 35324	= 5·5480699
9	379 |	6	74
6	25 | 3	3 | 5·5480773	
3	1 | 26		1·8493591
2 | 3·0146871			
	1·5073435		
add	1·8493591		
	3·3567026		
log 2273·5	= 3·3566950		
	76		
4	76		

Thus the product is 2273·54.

12. Let $a = 1{\cdot}0356270$ and $b = {\cdot}7503269$; then $a^2 - b^2 = (a+b)(a-b)$, and $a+b = 1{\cdot}7859539$, $a-b = {\cdot}2853001$.

log 1·7859	= ·2518571 |	
5	122 |	
3	7 | 3	
9	2 | 19	
log ·28530	= $\bar{1}$·4553018 |	
0	0 |	
1	1 | 5	
	$\bar{1}$·7071722	
log ·50953	$\bar{1}$·7071698	
	24	
2	17	
	70	
8	69	

Thus the difference is ·5095328.

13. $\log x = \frac{3}{5}\log 34{\cdot}7326 + \frac{1}{6}\log 2{\cdot}53894 - \frac{1}{5}\log 4{\cdot}39682.$

$\log 2{\cdot}5389 = {\cdot}4046456$

4 | 68

6 | ·4046524

·0674421

add ·7958146

·8632567

$\log 7{\cdot}2988 = {\cdot}8632515$

52

9 | 54

$\log 4{\cdot}3968 = {\cdot}6431367$

2 | 20

·6431387

$\log 34{\cdot}732 = 1{\cdot}5407298$

6 | 75

1·5407373

3

4·6222119

subtract ·6431387

5 | 3·9790732

·7958146

Thus $x = 7{\cdot}29889$.

14. $\log {\cdot}0037258 = \bar{3}{\cdot}5712195$

1 | 12

6 | 70

9 | 105

$\bar{3}{\cdot}5712215$

add $\bar{1}{\cdot}7505167$

2 | $\bar{3}{\cdot}3217382$

$\bar{2}{\cdot}6608691$

$\log {\cdot}045800 = \bar{2}{\cdot}6608655$

36

3 | 29

70

7 | 67

$\log {\cdot}56301 = \bar{1}{\cdot}7505161$

0 | 0

7 | 54

8 | 62

$\bar{1}{\cdot}7505167$

Thus the mean proportional is ·04580037.

15. If x be the required number, we have $x = \frac{{\cdot}03751786}{({\cdot}43607528)^2}$,

$\log {\cdot}037517 = \bar{2}{\cdot}5742281$

8 | 93

6 | 70

$\bar{2}{\cdot}5742381$

$\bar{1}{\cdot}2791230$

$\bar{1}{\cdot}2951151$

$\log {\cdot}19729 \quad \bar{1}{\cdot}2951051$

100

4 | 88

120

5 | 110

$\log {\cdot}43607 = \bar{1}{\cdot}6395562$

5 | 50

2 | 20

8 | 79

$\bar{1}{\cdot}6395615$

2

$\bar{1}{\cdot}2791230$

Thus $x = {\cdot}1972945$.

16. If x be the required number, we have

$$x = \frac{29{\cdot}302564 \times {\cdot}33025107}{56712{\cdot}43}.$$

log 29·302	= 1·4668973			log 56712	= 4·7536750		
5	74			4	31		
6	8	9		3	2	3	
4		59			4·7536783		
log ·33025	= $\bar{1}$·5188428						
1	13						
0	0						
7		92					
	·9857498						
subtract	4·7536783						
	$\bar{4}$·2320715						
log ·00017063	= $\bar{4}$·2320554						
	161						
6	152						
	90						
3	76						

Thus the fourth proportional is ·0001706363.

17. Let x be the required number, then

$$x = \sqrt[8]{(\cdot 035689)^{\frac{3}{5}} \times (2{\cdot}879432)^{\frac{5}{7}}}.$$

log 2·8794	= ·4593020			$\frac{1}{5}$ log ·035689	= $\frac{1}{5}$ ($\bar{2}$·5525344)		
3	45				= $\bar{1}$·7105069		
2	30						
	·4593068						
	3						
14	$\bar{1}$·3779204						
	·0984229						
	$\bar{1}$·7105069						
	$\bar{1}$·8089298						
log ·64406	= $\bar{1}$·8089263						
	35						
5	34						

Thus the geometric mean is ·644065.

18. Here $x = \dfrac{(7836{\cdot}43)^{\frac{1}{4}} \times (357{\cdot}814)^{\frac{1}{5}}}{(32{\cdot}7812)^{\frac{1}{3}}}$.

$\log 7836{\cdot}4 = 3{\cdot}8941166$

$3 \quad 17$

$4 \mid 3{\cdot}8941183$

$\cdot 9735295$

add $\cdot 5107315$

$1{\cdot}4842610$

subtract $\cdot 5052083$

$\cdot 9790527$

$\log 9{\cdot}5291 = \cdot 9790519$

8

$2 \quad 9$.

$\log 32{\cdot}781 = 1{\cdot}5156222$

$2 \quad 26$

$3 \mid 1{\cdot}5156248$

$\cdot 5052083$

$\log 357{\cdot}81 = 2{\cdot}5536525$

$4 \quad 49$

$5 \mid 2{\cdot}5536574$

$\cdot 5107315$

Thus the fourth proportional is 9·52912.

19. $\log \sin 27° 13' = \bar{1}{\cdot}6602550$

$\frac{12}{60} \times 2455 = 491$

$\bar{1}{\cdot}6603041$

$\bar{1}{\cdot}8414768$

$\bar{1}{\cdot}5017809$

$\log {\cdot}31752 = \bar{1}{\cdot}5017711$

98

$7 \quad 96$

20

$1 \quad 14$

$\log \cos 46° 2' = \bar{1}{\cdot}8415095$

subtract $\frac{15}{60} \times 1310 = 327$

$\bar{1}{\cdot}8414768$

Thus the required value is ·3175271.

20. $\cot 97° 14' 16'' = -\cot 82° 45' 44''$,

$\sec 112° 13' 5'' = -\sec 67° 46' 55''$.

$\log \sec 67° 46' = \cdot 4220725$

$\frac{11}{12} \times 3092 = 2834$

$\cdot 4223559$

$\bar{1}{\cdot}1038011$

$\bar{1}{\cdot}5261570$

$\log {\cdot}33585 = \bar{1}{\cdot}5261454$

116

$9 \quad 116$

$\log \cot 82° 45' = \bar{1}{\cdot}1045420$

$\frac{11}{15} \times 10103 = 7409$

$\bar{1}{\cdot}1038011$

Thus the required value is ·335859.

21.

log sin 20° 13′	$=\bar{1}\cdot5385375$	log cot 47° 53′	$=\bar{1}\cdot9562154$
$\frac{20}{60}\times3429$	$=1143$	*subtract* $\frac{15}{60}\times2540$	$=635$
log sec 42° 15′	$=\cdot1306403$		$\bar{1}\cdot9561519$
$\frac{30}{60}\times1148$	$=574$	add	$\bar{1}\cdot6693495$
	$\bar{1}\cdot6693495$		$1\cdot6255014$
		log ·42218	$=1\cdot6254977$
			37
		3	31
			60
		6	62

Thus the required value is ·4221836.

22.

log 324·13	$=2\cdot5107192$	log sin 113° 14′ 16″	
6	80		$=\log\sin 66°\,45'\,44''$,
8	10 7		
log 417·24	$=2\cdot6203859$	log sin 66° 45′	$=\bar{1}\cdot9632168$
3	31	$\frac{44}{60}\times543$	$=398$
1	1 0		$\bar{1}\cdot9632566$
	$5\cdot1311174$		
	$\bar{1}\cdot9632566$		
	$5\cdot0943740$		
log 12427	$=5\cdot0943663$		
	77		
2	70		
	70		
2	70		

Thus the required value is 12427·2.

23. Here $a=\dfrac{b\sin A}{\sin B}$, and $\sin B=\sin 60°\,45'\,42''$.

log sin 35° 15′	$=\bar{1}\cdot7612851$	log sin 60° 45′	$=\bar{1}\cdot9407634$
$\frac{33}{60}\times1787$	$=983$	$\frac{42}{60}\times708$	$=496$
	$\bar{1}\cdot7613834$		$\bar{1}\cdot9408130$
	$\bar{1}\cdot9408130$		
	$\bar{1}\cdot8205704$		
log 378·25	$=2\cdot5777789$		
	$2\cdot3983493$		
log 250·23	$=2\cdot3983394$		
	99		
5	87		
	120		
7	121		

Thus $a=250\cdot2357$.

24. (1) $\log \tan\theta = \frac{1}{3}(\log 5 - \log 12)$

$\log 5 = \quad \cdot 6989700$

$\log 12 = \quad 1\cdot 0791812$

$3 \mid \overline{1}\cdot 6197888$

$\log \tan\theta = \quad \overline{1}\cdot 8732629$

$\log \tan 36° 45' = \quad \overline{1}\cdot 8731668$

961

$\frac{961}{2634} \times 60'' = 22''.$

$\therefore\ \theta = 36° 45' 22''.$

(2) $(3\sin\theta - 1)(\sin\theta + 1) = 0.$

$\therefore\ \sin\theta = \frac{1}{3}$ or $-1.$

$\log \sin\theta = -\log 3$

$= \overline{1}\cdot 5228787$

$\log \sin 19° 28' = \overline{1}\cdot 5227811$

976

$\frac{976}{3572} \times 60'' = 16''.$

$\therefore\ \theta = 19° 28' 16''.$

25. $x = \sin 23° 18' 5'' \times \cot 38° 15' 13'' \times \cos 28° 17' 25''.$

$\log \cot 38° 15' = \quad \cdot 1032884$

subtract $\frac{13}{60} \times 2598 = \quad 562$

$\cdot 1032322$

$\overline{1}\cdot 9447579$

$\log \sin 23° 18' = \overline{1}\cdot 5971965$

$\frac{5}{60} \times 2932 = \quad 244$

$\overline{1}\cdot 6452110$

$\log \cdot 44178 = \overline{1}\cdot 6452061$

49

$5 \qquad 49$

$\log \cos 28° 17' = \overline{1}\cdot 9447862$

subtract $\frac{25}{60} \times 680 = \quad 283$

$\overline{1}\cdot 9447579$

Thus $x = \cdot 441785.$

26.

$\log \cos 32° 47' = \quad \overline{1}\cdot 9246535$

2

$\overline{1}\cdot 8493070$

$\log \cot 41° 19' = \quad \cdot 0559928$

$3 \mid \overline{1}\cdot 9052998$

$\overline{1}\cdot 9684333$

$\log \sin 68° 25' = \quad \overline{1}\cdot 9684286$

47

$\frac{47}{499} \times 60'' = 6''.$

Thus $\theta = 68° 25' 6''.$

EXAMPLES. XVI. a. PAGE 166.

1. First side $= \frac{s(s-a)}{c} + \frac{s(s-b)}{c} = \frac{s(2s-a-b)}{c} = s.$

2. First side $= s\sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \cdot \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$= s\sqrt{\frac{(s-a)^2}{s^2}} = s-a.$

3. First side $= \frac{1-\cos A}{1+\cos B} = \frac{2\sin^2\frac{A}{2}}{2\sin^2\frac{B}{2}} = \frac{(s-b)(s-c)}{bc} \times \frac{ca}{(s-c)(s-a)}$

$= \frac{a(s-b)}{c(s-a)} = \frac{a(a+c-b)}{b(b+c-a)}.$

4. First side $= \frac{b(s-b)(s-c)}{bc} + \frac{a(s-c)(s-a)}{ca}$

$= \frac{(s-c)\{s-b+s-a\}}{c} = \frac{c(s-c)}{c} = s-c.$

5. Each of the expressions reduces to $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

6. $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \sqrt{\frac{3\times 14}{24\times 7}} = \frac{1}{2}.$

7. $\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{21\times 6}{8\times 7}} = \frac{3}{2}.$

8. First side $= \frac{s(s-a)+s(s-b)+s(s-c)}{abc}$

$= \frac{s\{3s-(a+b+c)\}}{abc} = \frac{s^2}{abc}.$

9. First side $= \frac{b-c}{a} \cdot \frac{s(s-a)}{bc}$ + two similar terms

$= \frac{(b-c)(s^2-as)}{abc}$ + two similar terms

$= \frac{s^2\{(b-c)+(c-a)+(a-b)\} - s\{a(b-c)+b(c-a)+c(a-b)\}}{abc}$

$= 0.$

EXAMPLES. XVI. b. PAGE 169.

1. $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{7\times4}{5\times8}} = \sqrt{\frac{7}{10}}$.

$$\log\sin\frac{C}{2} = \frac{1}{2}(\log 7 - 1)$$
$$= \bar{1}\cdot9225490$$
$$\log\sin 56°\,47' = \bar{1}\cdot9225205$$
$$\text{diff.} \qquad\qquad 285$$

$$\text{prop}^{\text{l}}.\ \text{increase} = \frac{285}{827}\times 60'' = 20\cdot6'';$$

$$\begin{array}{r} 285 \\ 60 \\ \hline 827\,)\,17100\,(\,20\cdot6 \\ 1654 \\ \hline 5600 \\ 4962 \\ \hline \end{array}$$

$\therefore \frac{C}{2} = 56°\,47'\,20\cdot6''$, and $C = 113°\,34'\,41''$.

2. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{16\times24}{67\times27}} = \sqrt{\frac{128}{603}}$.

$$\log\tan\frac{A}{2} = \bar{1}\cdot6634464$$
$$= \log\tan 24°\,44'\,13'';$$
$$\therefore A = 49°\,28'\,26''.$$

$$\log 128 = 2\cdot1072100$$
$$\log 603 = 2\cdot7803173$$
$$2)\,\bar{1}\cdot3268927$$
$$\bar{1}\cdot6634464$$

3. $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} = \sqrt{\frac{15\times5}{2\times2\times4\times6}} = \frac{5}{4\sqrt{2}}$.

$$\log\cos\frac{B}{2} = \log 5 - \frac{5}{2}\log 2$$
$$= \bar{1}\cdot9463950$$
$$\log\cos 27°\,53' = \bar{1}\cdot9464040$$
$$\text{diff.} \qquad\qquad 90$$

$$\text{prop}^{\text{l}}.\ \textit{increase} = \frac{90}{669}\times 60'' = 8\cdot07'';$$

$$\log 5 = \cdot6989700$$
$$\frac{5}{2}\log 2 = \cdot7525750$$
$$\bar{1}\cdot9463950$$

$$\begin{array}{r} 30 \\ 60 \\ \hline 223\,)\,1800\,(\,8\cdot07 \\ 1784 \\ \hline 1600 \end{array}$$

$\therefore \frac{B}{2} = 27°\,53'\,8\cdot07''$, and $B = 55°\,46'\,16''$.

4. $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}=\sqrt{\frac{9\times 2}{5\times 6}}=\sqrt{\frac{6}{10}}$.

$$\log\cos\frac{C}{2}=\frac{1}{2}(\log 6-1)$$
$$=\bar{1}\cdot 8890757$$
$$\log\cos 39^\circ 14'=\bar{1}\cdot 8890644$$
$$\text{diff.}\qquad 113$$

$$\text{prop}^{\text{l}}.\ \textit{decrease}=\frac{113}{1032}\times 60''=6\cdot 6'';$$

$$\therefore\ \frac{C}{2}=39^\circ 13' 53\cdot 4'',\ \text{and}\ C=78^\circ 27' 47''.$$

$$1032\overline{)\,67800\,(\,6\cdot 6}\quad \begin{array}{r}113\\60\\ \\6192\\5880\\6192\end{array}$$

5. $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\sqrt{\frac{\cdot 75\times 2}{3\cdot 75}}=\sqrt{\frac{2^2}{10}}$.

$$\log\tan\frac{C}{2}=\frac{1}{2}\{2\log 2-1\}=\frac{1}{2}(\bar{1}\cdot 6020600)$$
$$=\bar{1}\cdot 8010300$$
$$\log\tan 32^\circ 18'=\bar{1}\cdot 8008365$$
$$\text{diff.}\qquad 1935$$

$$\text{prop}^{\text{l}}.\ \text{increase}=\frac{1935}{2796}\times 60''=41\cdot 5'';$$

$$\therefore\ \frac{C}{2}=32^\circ 18' 41\cdot 5'',\ \text{and}\ C=64^\circ 37' 23''.$$

$$2796\overline{)\,116100\,(\,41\cdot 5}\quad \begin{array}{r}1935\\60\\ \\11184\\4260\\2796\\14640\end{array}$$

6. $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\sqrt{\frac{3\times 4}{15\times 8}}=\sqrt{\frac{1}{10}}$.

$$\log\tan\frac{C}{2}=-\frac{1}{2}=\bar{1}\cdot 500000$$
$$\log\tan 17^\circ 33'=\bar{1}\cdot 500042$$
$$\text{diff.}\qquad 42$$

$$\text{prop}^{\text{l}}.\ \text{decrease}=\frac{42}{439}\times 60''=5\cdot 7'';$$

$$\therefore\ \frac{C}{2}=17^\circ 32' 54\cdot 3'',\ \text{and}\ C=35^\circ 5' 49''.$$

$$439\overline{)\,2520\,(\,5\cdot 7}\quad \begin{array}{r}42\\60\\ \\2195\\3250\end{array}$$

7. Let $a=4$, $b=10$, $c=11$.

$$\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}=\sqrt{\frac{25}{2}\cdot\frac{3}{2}\cdot\frac{1}{40}}=\sqrt{\frac{15}{2^5}}=\sqrt{\frac{30}{2^6}}.$$

$$\log\cos\frac{C}{2}=\frac{1}{2}(\log 3+1-6\log 2)$$

$$=\bar{1}\cdot 8354707$$

$\log\cos 46°\,47'=$	$\bar{1}\cdot 8355378$
diff.	671

prop^{l}. *increase* $=\frac{671}{1345}\times 60''=30''$.

$1+\log 3$	$=1\cdot 4771213$
$6\log 2$	$=1\cdot 8061800$
	$2)\,\bar{1}\cdot 6709413$
	$\bar{1}\cdot 8354707$

$\therefore\ \frac{C}{2}=46°\,47'\,30''$, and $C=93°\,35'$.

8. $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{63\times 7}{21\times 8}}=\frac{1}{2}$.

$$\log\tan\frac{B}{2}=-\log 2=\bar{1}\cdot 6989700$$

$\log\tan 26°\,33'=$	$\bar{1}\cdot 6986847$
diff.	2853

prop^{l}. increase $=\frac{2853}{3159}$ of $60''=54{\cdot}2''$.

$\therefore\ \frac{B}{2}=26°\,33'\,54{\cdot}2''$, and $B=53°\,7'\,48''$.

$$\begin{array}{r}2853\\60\\\hline 3159\,)\,171180\,(\,54{\cdot}2\\15795\\\hline 13230\\12636\\\hline 5940\end{array}$$

Again $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\sqrt{\frac{6\times 8}{21\times 7}}=\frac{4}{7}$.

$$\log\tan\frac{C}{2}=2\log 2-\log 7$$

$$=\bar{1}\cdot 7569620$$

$\log\tan 29°\,44'=$	$\bar{1}\cdot 7567587$
diff.	2033

prop^{l}. increase $=\frac{2033}{2933}\times 60''=41{\cdot}5$.

$\therefore\ \frac{C}{2}=29°\,44'\,41{\cdot}5''$, and $C=59°\,29'\,23''$.

$\therefore\ A=67°\,22'\,49''$.

$2\log 2$	$=\cdot 6020600$
$\log 7$	$=\cdot 8450980$
	$\bar{1}\cdot 7569620$

$$\begin{array}{r}2033\\60\\\hline 2933\,)\,121980\,(\,41{\cdot}5\\11732\\\hline 4660\\2933\\\hline 17270\end{array}$$

9. $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{3}{2}\times\frac{5}{2}\times\frac{2}{9}\times 2}=\sqrt{\frac{5}{3}}$.

$$\log\tan\frac{B}{2}=\frac{1}{2}(\log 5-\log 3)$$
$$=\cdot 1109244$$

$\log\tan 52°\,14'$	$=\cdot 1108395$
diff.	849

$\log 5 = \cdot 6989700$
$\log 3 = \cdot 4771213$
$2\,\big|\,\cdot 2218487$
$\cdot 1109244$

prop^{l}. increase $=\frac{849}{435}\times 10''=19\cdot 5''$;

$$\therefore\ \frac{B}{2}=52°\,14'\,19\cdot 5'',\ \text{and}\ B=104°\,28'\,39''.$$

Again $\tan\frac{C}{2}=\sqrt{\frac{(s-b)(s-a)}{s(s-c)}}=\sqrt{\frac{1}{2}\times\frac{3}{2}\times\frac{2}{9}\times\frac{2}{5}}=\frac{1}{\sqrt{3\times 5}}$.

$$\log\tan\frac{C}{2}=-\frac{1}{2}(\log 3+\log 5)$$
$$=\bar{1}\cdot 4119544$$

$\log\tan 14°\,28'$	$=\bar{1}\cdot 4116146$
diff.	3398

prop^{l}. increase $=\frac{3398}{870}\times 10''=39'$;

$$\therefore\ \frac{C}{2}=14°\,28'\,39'',\ \text{and}\ C=28°\,57'\,18''.$$

$$\therefore\ A=46°\,34'\,3''.$$

EXAMPLES. XVI. c. PAGE 173.

1. $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{1}{5}\cot 30°=\frac{2}{10}\sqrt{3}$.

$$\log\tan\frac{A-B}{2}=\log 2+\frac{1}{2}\log 3-1$$
$$=\bar{1}\cdot 5395907$$

$\log\tan 19°\,6'$	$=\bar{1}\cdot 5394287$
diff.	1620

$\log 2 = \cdot 3010300$
$\frac{1}{2}\log 3 = \cdot 2385607$
$\cdot 5395907$

prop^{l}. increase $=\frac{1620}{4084}\times 60''=24''$.

$$\therefore \frac{A-B}{2} = 19° 6' 24'', \text{ and } \frac{A+B}{2} = 60°;$$

$$\therefore A = 79° 6' 24''; \; B = 40° 53' 36''.$$

2. $$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2} = \frac{8}{10}\cot 32° 30'.$$

$$\log\tan\frac{C-A}{2} = 3\log 2 - 1 + \log\cot 32° 30'$$

$$= \cdot 0989027$$

$$\log\tan 51° 28' = \cdot 0988763$$

$$\text{diff.} \quad 264$$

$$\text{prop}^{\text{l}}.\ \text{increase} = \frac{264}{2592} \times 60'' = 6'';$$

$3\log 2 =$	$\cdot 9030900$
$\log\cot 32° 30' =$	$\cdot 1958127$
	$1 \cdot 0989027$

$$\therefore \frac{C-A}{2} = 51° 28' 6'',$$

$$\frac{C+A}{2} = 57° 33';$$

$$\therefore C = 108° 58' 6''; \; A = 6° 1' 54''.$$

3. $$\tan\frac{B-A}{2} = \frac{b-a}{b+a}\cot\frac{C}{2} = \frac{5}{12}\sqrt{3}.$$

$$\log\tan\frac{B-A}{2} = 1 - 3\log 2 - \frac{1}{2}\log 3$$

$$= \bar{1} \cdot 8583494$$

$$\log\tan 35° 49' = \bar{1} \cdot 8583357$$

$$\text{diff.} \quad 137$$

$\frac{1}{2}\log 3 =$	$\cdot 2385606$
$3\log 2 =$	$\cdot 9030900$
	$\bar{1} \cdot 1416506$

$$\text{prop}^{\text{l}}.\ \text{increase} = \frac{137}{2662} \times 10'' = \cdot 5'';$$

$$\therefore \frac{B-A}{2} = 35° 49' 0 \cdot 5'',$$

$$\frac{B+A}{2} = 60°;$$

$$\therefore B = 95° 49' 1''; \; A = 24° 10' 59''.$$

4. $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{4}{50}\cot 22°\,15'=\frac{8}{100}\cot 22°\,15'$.

$\log\tan\frac{B-C}{2}=\bar{1}\cdot2912491$

$\log\tan 11°\,3'=\bar{1}\cdot2906713$

diff. 5778

prop^{l}. increase $=\frac{5778}{6711}\times 60''=52''$;

$3\log 2-2=\bar{2}\cdot9030900$

$\log\cot 22°\,15'=\cdot3881591$

$\bar{1}\cdot2912491$

$\therefore\ \frac{B-C}{2}=11°\,3'\,52''$, and $\frac{B+C}{2}=67°\,45'$;

$\therefore\ B=78°\,48'\,52''$; $C=56°\,41'\,8''$.

5. $\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}=\frac{10}{32}\cot 17°\,21'\,15''$.

$\log\tan\frac{C-A}{2}=1+\log\cot 17°\,21'\,15''-5\log 2$

$=1\cdot5051500-1\cdot5051500$

$=0$.

$\therefore\ \frac{C-A}{2}=45°$, and $\frac{C+A}{2}=72°\,38'\,45''$;

$\therefore\ C=117°\,38'\,45''$; $A=27°\,38'\,45''$.

6. $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{1}{4}\cot 30°\,15'$.

$\log\tan\frac{A-B}{2}=-2\log 2+\log\cot 30°\,15'$

$=\bar{1}\cdot63214$

$\log\tan 23°\,13'=\bar{1}\cdot63240$

diff. 26

$\log\cot 30°\,15'=\cdot23420$

$2\log 2=\cdot60206$

$\bar{1}\cdot63214$

prop^{l}. *decrease* $=\frac{26}{35}\times 60''=45''$;

$\therefore\ \frac{A-B}{2}=23°\,12'\,15''$, and $\frac{A+B}{2}=59°\,45'$;

$\therefore\ A=82°\,57'\,15''$; $B=36°\,32'\,45''$.

7. $\tan\frac{A-C}{2}=\frac{a-c}{a+c}\cot\frac{B}{2}=\frac{71}{283}\cot 28°\,14'.$

$\log\tan\frac{A-C}{2}=\bar{1}\cdot3556602$

$\log\tan 12°\,46'=\bar{1}\cdot3552267$

diff. $\quad 4335$

prop^{l}. increase $=\frac{4335}{5859}\times 60''=44''$;

$\log 71=1\cdot8512583$

$\log\cot\frac{B}{2}=\cdot2700705$

$2\cdot1213288$

$\log 583=2\cdot7656686$

$\bar{1}\cdot3556602$

$\therefore\ \frac{A-C}{2}=12°\,46'\,44''$, and $\frac{A+C}{2}=61°\,46'$;

$\therefore\ A=74°\,32'\,44''$; $C=48°\,59'\,16''$.

8. $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{3}{5}\cot 32°\,30'.$

$\log\tan\frac{B-C}{2}=\bar{1}\cdot9739640$

$\log\tan 43°\,18'=\bar{1}\cdot9742133$

diff. $\quad 2493$

prop^{l}. *decrease* $=\frac{2493}{2531}\times 60''=59''$;

$\log 3=\cdot4771213$

$\log 5=\cdot6989700$

$\bar{1}\cdot7781513$

$\log\cot 32°\,30'=\cdot1958127$

$\bar{1}\cdot9739640$

$\therefore\ \frac{B-C}{2}=43°\,17'\,1''$, and $\frac{B+C}{2}=57°\,30'$;

$\therefore\ B=100°\,47'\,1''$; $C=14°\,12'\,59''$.

9. Here $\cot\frac{A-B}{2}=\frac{a+b}{a-b}\tan\frac{C}{2}=2\tan\frac{C}{2}$.

$\therefore\ \log\cot\frac{A-B}{2}=\log 2+\log\tan 15°\,5'\,2{\cdot}5''$

$=\bar{1}\cdot7316236$

$\log\cot 61°\,41'=\bar{1}\cdot7314436$

diff. $\quad 1800$

prop^{l}. *decrease* $=\frac{1800}{504}\times 10''=35{\cdot}7''$;

$\log 2=\cdot3010300$

$\log\tan 15°\,5'=\bar{1}\cdot4305727$

$\frac{2{\cdot}5}{10}\times 838=209$

$\bar{1}\cdot7316236$

$\therefore\ \frac{A-B}{2}=61°\,40'\,24{\cdot}3''$, and $\frac{A+B}{2}=74°\,54'\,51{\cdot}5''$;

$\therefore\ A=136°\,35'\,21{\cdot}8''$; $B=13°\,14'\,33{\cdot}2''$.

EXAMPLES. XVI. d. PAGE 174.

1. Here $A = 180° - 114° 45' = 65° 15'$.

$$c = \frac{a \sin C}{\sin A} = \frac{100 \sin 54° 30'}{\sin 65° 15'}.$$

$\log c = 1{\cdot}9525317$
$= \log 89{\cdot}646162;$
$\therefore\ c = 89{\cdot}646162.$

$\log \sin 54° 30' = \bar{1}{\cdot}9106860$
$\log 100 \quad = 2$
$\bar{1}{\cdot}9106860$
$\log \sin 65° 15' = \bar{1}{\cdot}9581543$
$\bar{1}{\cdot}9525317$

2. $a = \frac{c \sin A}{\sin C} = \frac{270 \sin 55°}{\sin 60°} = 270 \sin 55° \times \frac{2}{\sqrt{3}}.$

$\therefore \log a = 1 + 3 \log 3 + \log \sin 55° - \frac{1}{2} \log 3 + \log 2$

$= 2{\cdot}4071977$
$\log 255{\cdot}38 = 2{\cdot}4071869$
diff. $\quad 108$

$\therefore$ prop^{l}. increase $= \frac{108}{170} \times {\cdot}01 = {\cdot}0064;$

$\therefore\ a = 255{\cdot}3864.$

$\log 270 = 2{\cdot}4313639$
$\log \sin 55 = \bar{1}{\cdot}9133645$
$\log 2 = \ {\cdot}3010300$
$2{\cdot}6457584$
$\frac{1}{2} \log 3 = \ {\cdot}2385607$
$2{\cdot}4071977$

3. $c = \frac{b \sin C}{\sin B} = \frac{100 \sin 62° 5'}{\sin 72° 14'}.$

$\log c = 1{\cdot}96749$
$= \log 92{\cdot}788;$
$\therefore\ c = 92{\cdot}788.$

$\log \sin 62° 5' = \bar{1}{\cdot}94627$
$\log 100 \quad = 2$
$\bar{1}{\cdot}94627$
$\log \sin 72° 14' = \bar{1}{\cdot}97878$
$\bar{1}{\cdot}96749$

4. Here $A = 180° - 148° 40' = 31° 20'$.

$$b = \frac{a \sin B}{\sin A} = \frac{102 \sin 70° 30'}{\sin 31° 20'}.$$

$\log b = 2{\cdot}267$
$= \log 185;$
$\therefore\ b = 185.$

$\log 102 \quad = 2{\cdot}009$
$\log \sin 70° 30' = \bar{1}{\cdot}974$
$\bar{1}{\cdot}983$
$\log \sin 31° 20' = \bar{1}{\cdot}716$
$2{\cdot}267$

Again $c = \dfrac{a \sin C}{\sin A}$,

$\log c = 2{\cdot}283$

$= \log 192$;

$\therefore\ c = 192.$

$\log 102 = 2{\cdot}009$

$\log \sin 78° 10' = \bar{1}{\cdot}990$

$\overline{1{\cdot}999}$

$\log \sin 31° 20' = \bar{1}{\cdot}716$

$\overline{2{\cdot}283}$

5. Here $c = \dfrac{a \sin C}{\sin A} = \dfrac{123}{\sqrt{2} \sin 15° 43'}$.

$\log c = 2{\cdot}5066124$

$\log 321{\cdot}10 = 2{\cdot}5066403$

diff. $\overline{279}$

prop^{l}. *decrease* $= \dfrac{279}{135} \times {\cdot}01 = {\cdot}02066.$

Thus $c = 321{\cdot}0793.$

$\frac{1}{2} \log 2 = {\cdot}1505150$

$\log \sin 15° 43' = \bar{1}{\cdot}4327777$

$\overline{\bar{1}{\cdot}5832927}$

$\log 123 = 2{\cdot}0899051$

$\overline{2{\cdot}5066124}$

6. $a = \dfrac{b \sin A}{\sin B} = \dfrac{1006{\cdot}62 \sin 44°}{\sin 66°}$.

$\log 1006{\cdot}62 = 3{\cdot}0028656$

$\log \sin 44° = \bar{1}{\cdot}8417713$

$\overline{2{\cdot}8446369}$

$\log \sin 66° = \bar{1}{\cdot}9607302$

$\log a = \overline{2{\cdot}8839067}$

$\therefore\ a = 765{\cdot}4321.$

$c = \dfrac{b \sin C}{\sin B} = \dfrac{1006{\cdot}62 \sin 70°}{\sin 66°}$.

$\log 1006{\cdot}62 = 3{\cdot}0028656$

$\log \sin 70° = \bar{1}{\cdot}9729858$

$\overline{2{\cdot}9758514}$

$\log \sin 66° = \bar{1}{\cdot}9607302$

$\log c = \overline{3{\cdot}0151212}$

$\therefore\ c = 1035{\cdot}43.$

7. Here A = supplement of $75° 45'$;

$\therefore\ b = \dfrac{1652 \sin 26° 30'}{\sin 75° 45'}$.

$\log b = 2{\cdot}8852436$

$\log 767{\cdot}80 = 2{\cdot}8852481$

diff. $\overline{45}$

$\log 1652 = 3{\cdot}2180100$

$\log \sin 26° 30' = \bar{1}{\cdot}6495274$

$\overline{2{\cdot}8675374}$

$\log \sin 73° 45' = \bar{1}{\cdot}9822938$

$\overline{2{\cdot}8852436}$

prop^{l}. *decrease* $= \dfrac{45}{57} \times {\cdot}01 = {\cdot}008$; $\therefore\ b = 767{\cdot}792.$

Again $c = \dfrac{1652 \sin 47° 15'}{\sin 73° 45'}$;

$\log c = 3{\cdot}1016030$

$\log 1263{\cdot}6 = 3{\cdot}1016096$

diff. $\overline{66}$

$\log 1652 = 3{\cdot}2180100$

$\log \sin 47° 15' = \bar{1}{\cdot}8658868$

$\overline{3{\cdot}0838968}$

$\log \sin 73° 45' = \bar{1}{\cdot}9822938$

$\overline{3{\cdot}1016030}$

prop^{l}. *decrease* $= \dfrac{66}{344} \times {\cdot}1 = {\cdot}019$; $\therefore\ c = 1263{\cdot}58.$

EXAMPLES. XVI. e. PAGE 176.

1. $\sin A = \dfrac{a \sin B}{b} = \dfrac{145}{178} \sin 41° 10'.$

$\log \sin A = \bar{1}\cdot 7293399,$

$\therefore A = 32° 25' 35''.$

$\log 145$	$= 2\cdot 1613680$
$\log \sin 41° 10'$	$= \bar{1}\cdot 8183919$
	$1\cdot 9797599$
$\log 178$	$= 2\cdot 2504200$
	$\bar{1}\cdot 7293399$

2. $\sin B = \dfrac{b}{a} \sin A = \dfrac{127}{85} \sin 26° 26'.$

$\log \sin B = \bar{1}\cdot 8228972,$

$\therefore B = 41° 41' 28'';$

and since a is $< b$, there is another value of B, namely,

$B = 138° 18' 32''.$

$\log 127$	$= 2\cdot 1038037$
$\log \sin 26° 26'$	$= \bar{1}\cdot 6485124$
	$\bar{1}\cdot 7523161$
$\log 85$	$= 1\cdot 9294189$
	$\bar{1}\cdot 8228972$

3. $\sin B = \dfrac{b}{c} \sin C = \dfrac{4}{5} \sin 45° = \dfrac{8}{10} \cdot \dfrac{1}{\sqrt{2}},$

$\log \sin B = 3 \log 2 - 1 - \dfrac{1}{2} \log 2.$

$\therefore \log \sin B = \bar{1}\cdot 7525750.$

$\therefore B = 34° 26'$

and $A = 100° 34'.$

$3 \log 2 =$	$\cdot 9030900$
$1 + \frac{1}{2} \log 2 =$	$1\cdot 1505150$
	$\bar{1}\cdot 7525750$

4. $\sin B = \dfrac{b}{a} \sin A = \dfrac{1706}{1405} \sin 40°.$

$\log \sin B = \bar{1}\cdot 8923702$

$\log \sin 51° 18' = \bar{1}\cdot 8923342$

diff. $\overline{\quad\quad 360}$

prop^{l}. increase $= \dfrac{360}{1012} \times 60'' = 21''.$

$\log 1706$	$= 3\cdot 2319790$
$\log \sin 40°$	$= \bar{1}\cdot 8080675$
	$3\cdot 0400465$
$\log 1405$	$= 3\cdot 1476763$
	$\bar{1}\cdot 8923702$

$\therefore B = 51° 18' 21''$; but since $a < b$, there is another value of B, namely, $128° 41' 39''$.

5. $\sin C = \frac{c}{b}\sin B = \frac{394}{573}\sin 112° 4' = \frac{394}{573}\cos 22° 4'$.

$\therefore \log \sin C = \bar{1}\cdot 8043030$

$\log \sin 39° 35' = \bar{1}\cdot 8042757$

diff. $\quad 273$

prop^{l}. increase $= \frac{273}{1527} \times 60'' = 11''$;

$\log 394 = 2\cdot 5954962$

$\log \sin 112° 4' = \bar{1}\cdot 9669614$

$2\cdot 5624576$

$\log 573 = 2\cdot 7581546$

$\bar{1}\cdot 8043030$

$\therefore C = 39° 35' 11''$; and $A = 28° 20' 49''$.

6. $\sin C = \frac{c}{b}\sin B = \frac{12}{8\cdot 4}\sin 37° 36' = \frac{1}{\cdot 7}\sin 37° 36'$.

$\log \sin C = \bar{1}\cdot 9403352$

$\log \sin 60° 39' = \bar{1}\cdot 9403381$

diff. $\quad 29$

prop^{l}. *decrease* $= \frac{29}{711} \times 60'' = 2\cdot 4''$.

$\log \sin 37° 36' = \bar{1}\cdot 7854332$

$\log \cdot 7 = \bar{1}\cdot 8450980$

$\bar{1}\cdot 9403352$

$\therefore C = 60° 38' 58''$; but since $b < c$, there is another value of C, namely, $119° 21' 2''$. Thus $A = 81° 45' 2''$, or $23° 2' 58''$.

7. (i) Here $\sin C = \frac{c \sin A}{a} = \frac{250}{125} \times \frac{1}{2} = 1$.

$\therefore C = 90°$, and there is no ambiguity.

(ii) Here $\sin C = \frac{250}{200} \times \frac{1}{2} = \frac{5}{8}$, and since $a < c$ there will be two values of C satisfying the data.

(iii) Here $\sin C = \frac{125}{200} \times \frac{1}{2} = \frac{5}{16}$; but since $a > c$ there is only one solution.

From (ii) we have $\log \sin C = \log 5 - \log 8 = \bar{1}\cdot 79588$.

$\therefore C = 38° 41'$, or $141° 19'$; and $A = 111° 19'$, or $8° 41'$.

Now in the obtuse-angled triangle we have

$$b = \frac{a \sin B}{\sin A} = \frac{200 \sin 8° 41'}{\sin 30°}.$$

$\log b = 1\cdot 7809601$

$\log 60\cdot 389 = 1\cdot 7809578$

diff. $\quad 23$

prop^{l}. increase $= \frac{23}{72} \times \cdot 001 = \cdot 0003$.

$\therefore b = 60\cdot 3893$.

$\log 200 = 2\cdot 3010300$

$\log \sin 8° 41' = \bar{1}\cdot 1789001$

$1\cdot 4799301$

$\log \sin 30° = \bar{1}\cdot 6989700$

$1\cdot 7809601$

EXAMPLES. XVI. f. PAGE 180.

1. $$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}=\sqrt{\frac{3\cdot4}{12\cdot5}}=\sqrt{\frac{1}{5}};$$

$$\therefore \log\tan\frac{B}{2}=-\frac{1}{2}\log 5=-\frac{1}{2}(1-\log 2)$$

$$=\bar{1}\cdot6505150$$

$$\log\tan 24^\circ 5' = \bar{1}\cdot6502809$$

$$\text{diff.} \qquad\qquad 2341$$

$$\text{prop}^{\text{l}}.\ \text{increase}=\frac{2341}{3390}\times 60''=41\cdot4'';$$

$$\therefore \frac{B}{2}=24^\circ 5' 41\cdot4'', \text{ and } B=48^\circ 11' 23''.$$

$$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\sqrt{\frac{4\cdot5}{12\cdot3}}=\sqrt{\frac{5}{3^2}};$$

$$\therefore \log\tan\frac{C}{2}=\frac{1}{2}(1-\log 2-2\log 3)$$

$$=\bar{1}\cdot8723637$$

$$\log\tan 36^\circ 41' = \bar{1}\cdot8721123$$

$$\text{diff.} \qquad\qquad 2514$$

$$\text{prop}^{\text{l}}.\ \text{increase}=\frac{2514}{2637}\times 60''=57\cdot2'';$$

$$\therefore \frac{C}{2}=36^\circ 41' 57\cdot2'', \text{ and } C=73^\circ 23' 54'';$$

$$\therefore A=58^\circ 24' 43''.$$

2. $$\cot\frac{A}{2}=\frac{b+c}{b-c}\tan\frac{B-C}{2}$$

$$=\frac{512}{162}\tan 12^\circ=\frac{256}{81}\tan 12^\circ.$$

$$\therefore \log\cot\frac{A}{2}=8\log 2-4\log 3+\log\tan 12^\circ$$

$$=\bar{1}\cdot8272293=\log\cot 56^\circ 6' 27'';$$

$$\therefore \frac{A}{2}=56^\circ 6' 27'', \text{ and } A=112^\circ 12' 54''.$$

$$\therefore B+C=67^\circ 47' 6'', \text{ and } B-C=24^\circ;$$

$$\therefore B=45^\circ 53' 33'', \text{ and } C=21^\circ 53' 33''.$$

3. $\sin A = \frac{2}{7}$, if A is the *less* of the two acute angles.

$$\log \sin A = \log 2 - \log 7$$
$$= \bar{1}\cdot 455932$$
$$\log \sin 14^\circ 11' = \bar{1}\cdot 455921$$
$$\text{diff.} \qquad 11$$

$$\text{prop}^{\text{l}}.\ \text{increase} = \frac{11}{110} \times 60'' = 6''.$$

$\therefore\ A = 14^\circ 11' 6''$;

$\therefore\ B = 90^\circ - 14^\circ 11' 6'' = 75^\circ 48' 54''$.

4. Here $a = 2183$, $A = 30^\circ 22'$, $B = 78^\circ 14'$, $C = 71^\circ 24'$.

$$b = \frac{a \sin B}{\sin A} = \frac{2183 \sin 78^\circ 14'}{\sin 30^\circ 22'}.$$

$$\log b = 3\cdot 6260817$$
$$\log 4227\cdot 4 = 3\cdot 6260733$$
$$\text{diff.} \qquad 84$$

$$\text{prop}^{\text{l}}.\ \text{increase} = \frac{84}{103} \times \cdot 1 = \cdot 0815;$$

$\therefore\ b = 4227\cdot 4815.$

$$\log 2183 = 3\cdot 3390537$$
$$\log \sin B = \bar{1}\cdot 9907766$$
$$3\cdot 3298303$$
$$\log \sin A = \bar{1}\cdot 7037486$$
$$3\cdot 6260817$$

5. $$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{9} \cot 11^\circ 10'.$$

$$\therefore\ \log \tan \frac{B-C}{2} = \log \cot 11^\circ 10' - 2 \log 3$$
$$= \cdot 70465 - \cdot 95424 = \bar{1}\cdot 75041.$$

$$\therefore\ \frac{B-C}{2} = 29^\circ 22' 26'', \text{ and } \frac{B+C}{2} = 78^\circ 50';$$

$$\therefore\ B = 108^\circ 12' 26''; \ C = 49^\circ 27' 34''.$$

Now $$a = \frac{c \sin A}{\sin C};$$

$$\therefore\ \log a = \log 2 + \log \sin 22^\circ 20' - \log \sin 49^\circ 27' 34''$$
$$= \cdot 30103 + \bar{1}\cdot 57977 - \bar{1}\cdot 88079$$
$$= \cdot 00001;$$

$\therefore\ a = 1$, approximately.

6. $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\cdot56234\cot 29°\,21'\,3''.$

Now $\log\cot 29°\,21'=\cdot250015$

$\log\cot 29°\,22''=\cdot249715$

diff. for $60''$ $\quad\overline{300}$

$\therefore$ prop^{l}. *decrease* for $3''=\frac{3}{60}\times 300=15;$

$\therefore\ \log\cot 29°\,21'\,3''=\cdot250000.$

$\therefore\ \log\tan\frac{A-B}{2}=\bar{1}\cdot75+\cdot25=0.$

$\therefore\ \tan\frac{A-B}{2}=1,$ so that $\frac{A-B}{2}=45°.$

Also $\frac{A+B}{2}=60°\,38'\,57'';$ whence $A=105°\,38'\,57'',\ B=15°\,38'\,57''.$

7. $\sin B=\frac{b\sin A}{a}=\frac{12\sin 30°}{9};$

$\therefore\ \log\sin B=1\cdot07918+\bar{1}\cdot69897-\cdot95424=\bar{1}\cdot82391;$

$\therefore\ B=41°\,48'\,39''$ or $138°\,11'\,21''$, both values being admissible since $a<b$.

$\therefore\ C=108°\,11'\,21''$ or $11°\,48'\,39''.$

Again $\quad c=\frac{b\sin C_1}{\sin B_1}=\frac{12\sin 108°\,11'\,21''}{\sin 41°\,48'\,39''};$

$\therefore\ \log c=1\cdot07918+\bar{1}\cdot97774-\bar{1}\cdot82391=1\cdot23301;$

$\therefore\ c=17\cdot1.$

Similarly from $c=\frac{b\sin C_2}{\sin B_2}$, we easily obtain $c=3\cdot68.$

8. $\tan\frac{C}{2}=\frac{a-b}{a+b}\cot\frac{A-B}{2}=\frac{1}{2}\cot 45°=\frac{1}{2};$

$\therefore\ \log\tan\frac{C}{2}=\log 1-\log 2=\bar{1}\cdot6989700$

$\log\tan 26°\,33'\quad=\bar{1}\cdot6986847$

diff. $\quad\overline{2853}$

$\therefore$ prop^{l}. increase $=\frac{2853}{3159}\times 60''=54\cdot2'';$

$\therefore\ \frac{C}{2}=26°\,33'\,54\cdot2'',$ and $C=53°\,7'\,48''.$

Hence $\frac{A+B}{2}=63°\,26'\,6''$, and $\frac{A-B}{2}=45°$;

$\therefore\ A=108°\,26'\,6''$, $B=18°\,26'\,6''$.

9. (1) Let $a=1404$, $b=960$, $A=32°\,15'$;

then $\sin B=\frac{b\sin A}{a}=\frac{80}{117\operatorname{cosec}32°\,15'}$;

$\therefore\ \log\sin B=1+3\log 2-(2\log 3+\log 13+\log\operatorname{cosec}32°\,15')$

$=\bar{1}\cdot5621316$, on reduction.

$\therefore\ B=21°\,23'$; $\therefore\ C=126°\,22'$.

(2) Let $a=1404$, $b=960$, $B=32°\,15'$;

then $\sin A=\frac{117}{80\operatorname{cosec}32°\,15'}$;

$\therefore\ \log\sin A=2\log 3+\log 13-(1+3\log 2+\log\operatorname{cosec}32°\,15')$

$=\bar{1}\cdot8923236$, on reduction.

$\therefore\ A=51°\,18'$, or $128°\,42'$ since the solution is ambiguous.

$\therefore\ C=96°\,27'$, or $19°\,3'$.

10. We have $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{1-\frac{c}{b}}{1+\frac{c}{b}}\cot\frac{A}{2}$

$=\frac{1-\cos\phi}{1+\cos\phi}\cot\frac{A}{2}=\tan^2\frac{\phi}{2}\cot\frac{A}{2}$,

where $\cos\phi=\frac{c}{b}=\frac{10}{11}$.

Hence $\log\cos\phi=1-\log 11=\bar{1}\cdot958607$;

$\therefore\ \phi=24°\,37'\,12''$.

Again $\log\tan\frac{B-C}{2}=2\log\tan\frac{\phi}{2}+\log\cot\frac{A}{2}$

$=\bar{2}\cdot677782+\cdot495800$

$=\bar{1}\cdot173582$.

$\therefore\ \frac{B-C}{2}=8°\,28'\,56\cdot5''$, and $\frac{B+C}{2}=72°\,17'\,30''$;

$\therefore\ B=80°\,46'\,26\cdot5''$, $C=63°\,48'\,33\cdot5''$.

11. $\sin B = \frac{b}{a}\sin A = \frac{1071}{873}\sin 50°$;

$\therefore \log\sin B = 3{\cdot}029789 + \bar{1}{\cdot}884254 - 2{\cdot}941014$

$= \bar{1}{\cdot}973029$

$\log\sin 70° =$	$\bar{1}{\cdot}972986$	$\log\sin 70°\ 1' =$	$\bar{1}{\cdot}973032$
$\log\sin 70° =$	$\bar{1}{\cdot}972986$
diff.	43	diff. for 1′	46

$$\text{prop}^{\text{l}}.\ \text{increase} = \frac{43}{46}\times 60'' = 56'';$$

$\therefore B = 70°\,0'\,56''$, or $109°\,59'\,4''$, both values being admissible since $a<b$.

12. As in Art. 195 we easily obtain

$$c^2 = (a+b)^2\sin^2\frac{C}{2}\left\{1+\left(\frac{a-b}{a+b}\right)^2\cot^2\frac{C}{2}\right\}.$$

Now let $$\frac{a-b}{a+b}\cot\frac{C}{2} = \tan\theta \quad \ldots\ldots(1),$$

then $$c = (a+b)\sin\frac{C}{2}\sec\theta \quad \ldots\ldots(2).$$

From (1), $$\tan\theta = \frac{1}{3}\cot 18°\,26'\,6'',$$

$$\therefore \log\tan\theta = {\cdot}47712 - {\cdot}47712 = 0.$$

Whence $$\tan\theta = 1, \text{ and } \theta = 45°.$$

From (2), $$c = 9\sqrt{2}\sin 18°\,26'\,6'';$$

$$\therefore \log c = 2\log 3 + \frac{1}{2}\log 2 + \log\sin 18°\,26'\,6''$$

$$= {\cdot}95424 + {\cdot}15052 + \bar{1}{\cdot}5$$

$$= {\cdot}60476.$$

$$\therefore c = 4{\cdot}0249.$$

13. Here $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$, and $s-a=549$, $s-c=291$;

$$\therefore \log\sin\frac{B}{2} = \frac{1}{2}(\log 549 + \log 291 - \log 1000 - \log 1258).$$

$\log 549$	$= 2{\cdot}7395723$	$\log 1000$	$= 3$
$\log 291$	$= 2{\cdot}4638930$	$\log 1258$	$= 3{\cdot}0996806$
	$\bar{5}{\cdot}2034653$		$6{\cdot}0996806$
	$6{\cdot}0996806$		
	$2\,)\ \bar{1}{\cdot}1037847$		
$\log\sin\frac{B}{2} =$	$\bar{1}{\cdot}5518924$	Diff. for $60'' = 3313$;	
$\log\sin 20°\,52' =$	$\bar{1}{\cdot}5516871$	$\therefore$ prop$^{\text{l}}$. increase $= \frac{2053}{3313}\times 60'' = 37''$.	
diff.	2053		

$$\therefore \frac{B}{2} = 20°\,52'\,37'', \text{ and } B = 41°\,45'\,14''.$$

14. $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{326\times199}{976\times451}}$.

$\log 326$	=	2·5132176	$\log 976 = 2\cdot9894498$
$\log 199$	=	2·2988531	$\log 451 = 2\cdot6541765$
		4·8120707	5·6436263
		5·6436263	
		$2)\ \bar{1}\cdot1684444$	
$\log\tan\frac{A}{2}$	=	$\bar{1}\cdot5842222$	
$\log\tan 21°\,0'$	=	$\bar{1}\cdot5841774$	Diff. for $20''=3775$;
diff.		448	$\therefore$ prop^{l}. increase $=\frac{448}{3775}\times60''=7''$;

$\therefore\ \frac{A}{2}=21°\,0'\,7''$, and $A=42°\,0'\,14''$.

Again $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{451\times199}{976\times326}}$;

$\log 199$	=	2·2988531	$\log 976 = 2\cdot9894498$
$\log 451$	=	2·6541765	$\log 326 = 2\cdot5132176$
		4·9530296	5·5026674
		5·5026674	
		$2)\ \bar{1}\cdot4503622$	
$\log\tan\frac{B}{2}$	=	$\bar{1}\cdot7251811$	
$\log\tan 27°\,58'$	=	$\bar{1}\cdot7250646$	Diff. for $60''=3\cdot049$;
diff.		1165	$\therefore$ prop^{l}. increase $=\frac{1165}{3049}\times60''=23''$;

$\therefore\ \frac{B}{2}=27°\,58'\,23''$, and $B=55°\,56'\,46''$.

$\therefore\ C=82°\,3'$.

15. $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}=\sqrt{\frac{2\frac{1}{2}\times1\frac{1}{2}}{5\times6}}=\sqrt{\frac{1}{8}}$;

$\therefore\ \log\sin\frac{A}{2}=-\frac{3}{2}\log 2$ Diff. for $1'=3342$;

$=\bar{1}\cdot5484550$, $\therefore$ prop^{l}. increase $=\frac{965}{3342}\times60''=17\cdot3''$;

$\log\sin 20°\,42'=\bar{1}\cdot5483585$

diff. 965

$\therefore\ A=41°\,24'\,35''$.

16. Here $\sin C = \frac{c}{b} = \frac{28{\cdot}58}{57{\cdot}321}$.

$\log 28{\cdot}58 = 1{\cdot}4560622$
$\log 57{\cdot}321 = 1{\cdot}7583138$
$\bar{1}{\cdot}6977484$
$\log \sin 29^\circ 54' = \bar{1}{\cdot}6976545$
diff. 939

Diff. for $60'' = 2196$;

$\therefore$ prop^{l}. increase $= \frac{939}{2196} \times 60'' = 26''$.

$\therefore C = 29^\circ\ 54'\ 26''$; whence $A = 60^\circ\ 5'\ 34''$.

17. Let C be the right angle, and $A = 18^\circ\ 37'\ 29''$; then

$$c = \frac{a}{\sin A} = \frac{284}{\sin 18^\circ\ 37'\ 29''}.$$

$\log 284 = 2{\cdot}4533183$
$\log \sin 18^\circ\ 37'\ 29'' = \bar{1}{\cdot}5042917$
$\log c = 2{\cdot}9490266$
$\log 889{\cdot}25 = 2{\cdot}9490239$
27
$5 \quad 25$
20
$4 \quad 20$

$\log \sin 18^\circ 37' = \bar{1}{\cdot}5041105$
$\frac{29}{60''} \times 8748 = 1812$
$\log \sin 18^\circ\ 37'\ 29'' = \bar{1}{\cdot}5042917$

$\therefore c = 889{\cdot}2554$ feet.

18. $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{1}{8} \cdot \sqrt{3}$;

$\therefore \log \tan \frac{B-C}{2} = \frac{1}{2} \log 3 - 3 \log 2$
$= \bar{1}{\cdot}3354706$
$\log \tan 12^\circ\ 12' = \bar{1}{\cdot}3348711$
diff. 5995

Diff. for $60'' = 6112$;

$\therefore$ prop^{l}. increase $= \frac{5995}{6112} \times 60'' = 59''$.

$\therefore \frac{B-C}{2} = 12^\circ\ 12'\ 59''$, and $\frac{B+C}{2} = 60^\circ$;

$\therefore B = 72^\circ\ 12'\ 59''$, $C = 47^\circ\ 47'\ 1''$.

19. Let AC be the ladder, C the window, and B the foot of the wall; then from the right-angled triangle ABC,

$$AC=b = \frac{42{\cdot}37}{\sin 72^\circ 15'},$$

$$\log b = 1{\cdot}6270585 - \bar{1}{\cdot}9788175$$

$$= 1{\cdot}6482410$$

$$\log 44{\cdot}487 = 1{\cdot}6482331$$

$$79$$

$$8 \qquad 78$$

$\therefore$ length of ladder $= 44{\cdot}4878$ feet.

20. $$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{9{\cdot}99}{53{\cdot}91}\cot 17^\circ 30'.$$

$\log 9{\cdot}99 = \cdot9995655$

$\log \cot 17^\circ 30' = \cdot5012777$

$1{\cdot}5008432$

$\log 53{\cdot}91 = 1{\cdot}7316693$

$\bar{1}{\cdot}7691739$

$\log \tan 30^\circ 26' = \bar{1}{\cdot}7689922$

diff. 1817

Diff. for $60'' = 2892$;

$\therefore$ propl. increase $= \frac{1817}{2892}\times 60'' = 38''$;

$$\therefore \frac{A-B}{2}=30^\circ 26' 38'', \text{ and } \frac{A+B}{2}=72^\circ 30';$$

$$\therefore A=102^\circ 56' 38'',\ B=42^\circ 3' 22''.$$

21. $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{11{\cdot}29}{38{\cdot}95}\cot 23^\circ 37' 30''.$$

$\log \cot 23^\circ 37' = \cdot3592844$

Subtract $\frac{30}{60}\times 3441$ $\quad 1721$

$\cdot3591123$

$\log 11{\cdot}29 = 1{\cdot}0526939$

$1{\cdot}4118062$

$\log 38{\cdot}95 = 1{\cdot}5905075$

$\log \tan \frac{B-C}{2} = \bar{1}{\cdot}8212987$

$\log \tan 33^\circ 31' = \bar{1}{\cdot}8210574$

diff. 2413

Diff. for $60'' = 2743$;

$\therefore$ propl. increase $= \frac{2413}{2743}\times 60'' = 53''$.

$$\therefore \frac{B-C}{2}=33^\circ 31' 53'', \text{ and } \frac{B+C}{2}=60^\circ 22' 30'';$$

$$\therefore B=99^\circ 54' 23'',\ C=32^\circ 50' 37''.$$

Again $a = \dfrac{b \sin A}{\sin B} = \dfrac{25{\cdot}12 \sin 47° 15'}{\sin 99° 54' 23''} = \dfrac{25{\cdot}12 \sin 47° 15'}{\cos 9° 54' 23''}$.

$\log \sin 47° 15'$	=	$\bar{1}{\cdot}8658868$	$\log \cos 9° 54' = \bar{1}{\cdot}9934844$
$\log 25{\cdot}12$	=	$1{\cdot}4000196$	*Subtract* $\frac{23}{60} \times 220 =$ 84
		$1{\cdot}2659064$	$\bar{1}{\cdot}9934760$
		$\bar{1}{\cdot}9934760$	
$\log a$	=	$\overline{1{\cdot}2724304}$	
$\log 18{\cdot}725$	=	$1{\cdot}2724218$	
		86	
4		93	$\therefore\ a = 18{\cdot}7254.$

22. $\sin \dfrac{B}{2} = \sqrt{\dfrac{(s-a)(s-c)}{ac}} = \sqrt{\dfrac{1361{\cdot}12 \times 1024{\cdot}48}{1837{\cdot}2 \times 2173{\cdot}84}}$.

$\log 1361{\cdot}1$	=	$3{\cdot}1338900$	$\log 1837{\cdot}2 = 3{\cdot}2641564$
2		64	$\log 2173{\cdot}8 = 3{\cdot}3372196$
$\log 1024{\cdot}4$	=	$3{\cdot}0104696$	4 80
8		339	$6{\cdot}6013840$
		$6{\cdot}1443999$	
		$6{\cdot}6013840$	
		$2)\ \bar{1}{\cdot}5430159$	
$\log \sin \frac{B}{2}$	=	$\bar{1}{\cdot}7715079$	Diff. for $60'' = 1724$;
$\log \sin 36° 13'$	=	$\bar{1}{\cdot}7714702$	$\therefore$ prop^{l}. increase $= \frac{377}{1724} \times 60'' = 13''$.
diff.		377	

$\therefore\ \frac{B}{2} = 36° 13' 13''$, and $B = 72° 26' 26''$.

23. $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\dfrac{6{\cdot}4405 \times 14{\cdot}9114}{52{\cdot}1248 \times 30{\cdot}7728}}$.

$\log 6{\cdot}4405$	=	$\cdot 8089196$	$\log 52{\cdot}124 = 1{\cdot}7170377$
$\log 14{\cdot}911$	=	$1{\cdot}1735068$	8 66
4		116	$\log 30{\cdot}772 = 1{\cdot}4881557$
		$\bar{1}{\cdot}9824380$	8 113
		$3{\cdot}2052113$	$3{\cdot}2052113$
		$2)\ \overline{2{\cdot}7772267}$	
$\log \tan \frac{A}{2}$	=	$\bar{1}{\cdot}3886134$	Diff. for $60'' = 5475$;
$\log \tan 13° 44'$	=	$\bar{1}{\cdot}3880837$	$\therefore$ prop^{l}. increase $= \frac{5297}{5475} \times 60'' = 58''$.
		5297	

$\therefore\ \frac{A}{2} = 13° 44' 58''$, and $A = 27° 29' 56''$.

Again $$\tan\frac{B}{2}=\sqrt{\frac{14\cdot9114\times30\cdot7728}{52\cdot1248\times\ 6\cdot4405}}.$$

log 14·9114	=	1·1735184	log 52·1248 = 1·7170443
log 30·7728	=	1·4881670	log 6·4405 = ·8089196
		2·6616854	2·5259639
		2·5259639	
	2)	·1357215	
$\log\tan\frac{B}{2}$	=	·0678608	Diff. for 60″ = 2558;
log tan 49° 27′	=	·0677338	∴ prop^{l}. increase $=\frac{1270}{2558}\times60''=30''$.
diff.		1270	

$$\therefore\ \frac{B}{2}=49^\circ\,27'\,30'',\text{ and } B=98^\circ\,55'.$$

$$\therefore\ C=53^\circ\,35'\,4''.$$

24. $$\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}=\frac{202\cdot949}{1497\cdot597}\cot51^\circ\,36'\,27''.$$

log cot 51° 36′	=	$\bar{1}$·8990487	log 1497·5 = 3·1753668
Subtract $\frac{27}{60}\times2595$	=	1168	9 261
		$\bar{1}$·8989319	7 203
log 202·94	=	2·3073677	3·1753949
9		193	
		2·2063189	
		3·1753949	
$\log\tan\frac{C-B}{2}$	=	$\bar{1}$·0309240	
log tan 6° 7′	=	$\bar{1}$·0300464	
diff.		8776	

Diff. for 60″ = 11909; ∴ prop^{l}. increase $=\frac{8776}{11909}\times60''=44''$;

$$\therefore\ \frac{C-B}{2}=6^\circ\,7'\,44'',\text{ and } \frac{C+B}{2}=38^\circ\,23'\,33'';$$

$$\therefore C=44^\circ\,31'\,17'',\ B=32^\circ\,15'\,49''.$$

To find a, we have $a = \dfrac{b \sin A}{\sin B} = \dfrac{647{\cdot}324 \sin 103° 12' 54''}{\sin 32° 15' 49''}$.

Now $\log \sin 103° 12' 54'' = \log \sin 76° 47' 6''$.

$\log \sin 76° 47' = \bar{1}{\cdot}9883415$

$\frac{6}{60} \times 297 = 30$

$\bar{1}{\cdot}9883445$

$\log 647{\cdot}32 = 2{\cdot}8111190$

$4 \quad 27$

$2{\cdot}7994662$

$\bar{1}{\cdot}7273911$

$\log a = \overline{3{\cdot}0720751}$

$\log 1180{\cdot}5 = \overline{3{\cdot}0720660}$

91

$2 \quad 73$

180

$5 \quad 185$

$\log \sin 32° 15' = \bar{1}{\cdot}7272276$

$\frac{49}{60} \times 2002 = 1635$

$\bar{1}{\cdot}7273911$

$\therefore \ a = 1180{\cdot}525.$

25. $a = \dfrac{b \sin A}{\sin B} = \dfrac{23{\cdot}2783 \sin 37° 57'}{\sin 43° 13'}$.

$\log 23{\cdot}278 = 1{\cdot}3669457$

$3 \quad 56$

$\log \sin 37° 57' = \bar{1}{\cdot}7888565$

$\bar{1}{\cdot}1558078$

$\log \sin 43° 13' = \bar{1}{\cdot}8355378$

$\log a = \overline{1{\cdot}3202700}$

$\log 20{\cdot}905 = 1{\cdot}3202502$

198

$9 \quad 187$

$\therefore \ a = 20{\cdot}9059.$

Again $$c = \frac{b \sin C}{\sin B} = \frac{23{\cdot}2783 \sin 81° 10'}{\sin 43° 13'}.$$

$$\begin{aligned} \log c &= 1{\cdot}3669503 + \bar{1}{\cdot}9948181 - \bar{1}{\cdot}8355378 \\ &= 1{\cdot}5262316 \\ \log 33{\cdot}591 &= 1{\cdot}5262229 \\ &\qquad\quad 87 \\ 7 &\qquad\quad 90 \end{aligned}$$

$$\therefore\ c = 33{\cdot}5917.$$

26. $$b = \frac{c \sin B}{\sin C} = \frac{2484{\cdot}3 \sin 72° 43' 25''}{\sin 47° 12' 17''}.$$

$$\begin{aligned} \log \sin 72° 43' &= \bar{1}{\cdot}9799339 \\ \tfrac{25}{60} \times 393 &= \quad 164 \\ \log 2484{\cdot}3 &= 3{\cdot}3952040 \\ &\quad\ 3{\cdot}3751543 \\ &\quad\ \bar{1}{\cdot}8655693 \\ \log b &= 3{\cdot}5095850 \\ \log 3232{\cdot}8 &= 3{\cdot}5095788 \\ &\qquad\quad 62 \\ 4 &\qquad\quad 54 \\ 6 &\qquad\quad 8 \end{aligned}$$

$$\begin{aligned} \log \sin 47° 12' &= \bar{1}{\cdot}8655362 \\ \tfrac{17}{60} \times 1169 &= \quad 331 \\ &\quad\ \bar{1}{\cdot}8655693 \end{aligned}$$

$$\therefore\ b = 3232{\cdot}846.$$

Again $$a = \frac{c \sin A}{\sin C} = \frac{2484{\cdot}3 \sin 60° 4' 18''}{\sin 47° 12' 17''}.$$

$$\begin{aligned} \log \sin 60° 4' &= \bar{1}{\cdot}9378220 \\ \tfrac{18}{60} \times 727 &= \quad 218 \\ \log 2484{\cdot}3 &= 3{\cdot}3952040 \\ &\quad\ 3{\cdot}3330478 \\ \log \sin 47° 12' 17'' &= \bar{1}{\cdot}8655693 \\ &\quad\ 3{\cdot}4674785 \\ \log 2934{\cdot}1 &= 3{\cdot}4674749 \\ &\qquad\quad 36 \\ 2 &\qquad\quad 30 \\ 4 &\qquad\quad 6 \end{aligned}$$

$$\therefore\ a = 2934{\cdot}124.$$

27. $\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}=\frac{4367}{4667}\cot 15°\,45'$.

$$\log\tan\frac{C-B}{2}=3\cdot6401832+\cdot5497060-3\cdot6690378$$

$$=\cdot5208514$$

$\log\tan 73°\,13'=\cdot5205681$ Diff. for $60''=4568$;

diff. 2833 $\therefore$ prop^{l}. increase $=\frac{2833}{4568}\times 60''=37''$.

$$\therefore\frac{C-B}{2}=73°\,13'\,37'',\text{ and }\frac{C+B}{2}=74°\,15';$$

$$\therefore C=147°\,28'\,37'',\ B=1°\,1'\,23''.$$

Again $$a=\frac{c\sin A}{\sin C}=\frac{4517\sin 31°\,30'}{\sin 32°\,31'\,23''}.$$

$\log 4517=3\cdot6548501$

$\log\sin 31°\,30'=\bar{1}\cdot7180851$

$3\cdot3729352$

$\log\sin 32°\,31'\,23''=\bar{1}\cdot7304907$

$\log a=3\cdot6424445$

$\log 4389\cdot8=3\cdot6424447$

$\log\sin 32°\,31'=\bar{1}\cdot7304148$

$\frac{23}{60}\times 1981=\quad 759$

$\log\sin 32°\,31'\,23''=\bar{1}\cdot7304907$

$$\therefore a=4389\cdot8\text{ nearly.}$$

28. $\sin A=\frac{a\sin C}{c}=\frac{324\cdot68\sin 35°\,17'\,12''}{421\cdot73}$.

$\log\sin 35°\,17'=\bar{1}\cdot7616424$

$\frac{12}{60}\times 1784=\quad 357$

$\log 324\cdot68=2\cdot5114555$

$2\cdot2731336$

$\log 421\cdot73=2\ 6250345$

$\log\sin A=\bar{1}\cdot6480991$

$\log\sin 26°\,24'=\bar{1}\cdot6480038$

diff. 953

Diff. for $60''=2544$;

$\therefore$ prop^{l}. increase $=\frac{953}{2544}\times 60''=23''$;

$$\therefore A=26°\,24'\,23'',\text{ and }\therefore B=118°\,18'\,25''.$$

Again $$b=\frac{c\sin B}{\sin C}=\frac{421{\cdot}73\sin 61°\,41'\,35''}{\sin 35°\,17'\,12''}.$$

$\log\sin 61°\,41' = \bar{1}{\cdot}9446501$

$\frac{35}{60}\times 680 = 397$

$\log 421{\cdot}73 = 2{\cdot}6250345$

$2{\cdot}5697243$

$\log\sin 35°\,17'\,12'' = \bar{1}{\cdot}7616781$

$2{\cdot}8080462$

$\log 642{\cdot}75 = 2{\cdot}8080421$

41

$6 \qquad 41$

$\therefore\ b=642{\cdot}756.$

29. $$\sin C=\frac{c\sin A}{a}=\frac{435{\cdot}6\sin 36°\,18'\,27''}{321{\cdot}7}.$$

$\log\sin 36°\,18' = \bar{1}{\cdot}7723314$

$\frac{27}{60}\times 1719 = 774$

$\log 435{\cdot}6 = 2{\cdot}6390879$

$2{\cdot}4114967$

$\log 321{\cdot}7 = 2{\cdot}5074511$

$\log\sin C = \bar{1}{\cdot}9040456$

$\log\sin 53°\,17' = \bar{1}{\cdot}9039587$

diff. 869

Diff. for $60'' = 943$;

$\therefore$ prop^{l}. increase $=\frac{869}{943}\times 60'' = 55''$;

$\therefore\ C=53°\,17'\,55''$, or $126°\,42'\,5''$, both values being admissible since $a<c$.

30. $$\sin C=\frac{c\sin B}{b}=\frac{1665}{1325}\sin 52°\,19'.$$

$\log\sin C = 3{\cdot}2214142 + \bar{1}{\cdot}8983968 - 3{\cdot}1222159$

$= \bar{1}{\cdot}9975951$

$\log\sin 83°\,58' = \bar{1}{\cdot}9975877$

diff. 74

Diff. for $60'' = 134$;

$\therefore$ prop^{l}. increase $=\frac{74}{134}\times 60'' = 33''$.

$\therefore\ C=83°\,58'\,33''$, or $96°\,1'\,27''$, both values being admissible since $b<c$

Now, with diagram of page 132, we have $\angle BAC_2 = 31° 39' 33''$.

$$\therefore a = \frac{b \sin A}{\sin B} = \frac{1325 \sin 31° 39' 33''}{\sin 52° 19'}.$$

$\log \sin 31° 39' = \bar{1}\cdot 7199350$

$\frac{33}{60} \times 2049 = 1127$

$\log 1325 = 3\cdot 1222159$

$2\cdot 8422636$

$\log \sin 52° 19' = \bar{1}\cdot 8983968$

$\log a = 2\cdot 9438668$

$\log 878\cdot 75 = 2\cdot 9438653$

15

3 15

$\therefore a = 878\cdot 753.$

31. Here $A = 64° 26' 15''$, $b = \frac{a \sin B}{\sin A} = \frac{3795 \sin 73° 15' 15''}{\sin 64° 26' 15''}$.

$\log \sin 73° 15' = \bar{1}\cdot 9811711$

$\frac{15}{60} \times 380 = 95$

$\log 3795 = 3\cdot 5792118$

$3\cdot 5603924$

$\log \sin 64° 26' 15'' = \bar{1}\cdot 9552620$

$\log 6 = 3\cdot 6051304$

$\log 4028\cdot 3 = 3\cdot 6051218$

86

8 86

$\log \sin 64° 26' = \bar{1}\cdot 9552469$

$\frac{15}{60} \times 604 = 151$

$\bar{1}\cdot 9552620$

$\therefore b = 4028\cdot 38.$

Again $c = \frac{a \sin C}{\sin A} = \frac{3795 \sin 42° 18' 30''}{\sin 64° 26' 15''}$.

$\log 3795 = 3\cdot 5792118$

$\log \sin 42° 18' = \bar{1}\cdot 8280231$

$\frac{30}{60} \times 1388 = 694$

$3\cdot 4073043$

$\log \sin A = \bar{1}\cdot 9552620$

$\log c = 3\cdot 4520423$

$\log 2831\cdot 6 = 3\cdot 4520319$

104

7 107

$\therefore c = 2831\cdot 67.$

32. $\sin B = \frac{b}{c}\sin C = \frac{17}{12}\sin 43° 12' 12''$.

$\log 17 = 1·2304489$

$\log \sin 43° 12' = \bar{1}·8354033$

$\frac{12}{60} \times 1345 = 269$

$1·0658791$

$\log 12 = 1·0791812$

$\log \sin B = \bar{1}·9866979$

$\log \sin 75° 53' = \bar{1}·9866827$

diff. 152

Diff. for $60'' = 317$;

$\therefore$ prop^{l}. increase $= \frac{152}{317} \times 60'' = 29''$.

$\therefore B = 75° 53' 29''$, or $104° 6' 31''$, both values being admissible since $c < b$.

$\therefore A = 60° 54' 19''$, or $32° 41' 17''$.

33. Let $b = 2·7402$, $c = ·7401$, $A = 59° 27' 5''$.

$$\tan \frac{B-C}{2} = \frac{2·0001}{3·4803} \cot 29° 43' 32·5''.$$

$\log \cot 29° 43' = ·2435347$

$\frac{32·5}{60} \times 2934 = 1589$

$\log 2·0001 = ·3010517$

$·5444275$

$\log 3·4803 = ·5416167$

$\log \tan \frac{B-C}{2} = ·0028108$

$\log \tan 45° 11' = ·0027793$

diff. 315

Diff. for $60'' = 2527$;

$\therefore$ prop^{l}. increase $= \frac{315}{2527} \times 60'' = 7·5''$.

$\therefore \frac{B-C}{2} = 45° 11' 7·5''$, and $\frac{B+C}{2} = 60° 16' 27·5''$;

$\therefore B = 105° 27' 35''$, $C = 15° 5' 20''$.

Again $a = \frac{c \sin A}{\sin C} = \frac{·7401 \sin 59° 27' 5''}{\sin 15° 5' 20''}$.

$\log \sin 59° 27' = \bar{1}·9350969$

$\frac{5}{60} \times 746 = 62$

$\log ·7401 = \bar{1}·8692904$

$\bar{1}·8043935$

$\bar{1}·4155029$

$\log a = ·3888906$

$\log 2·4484 = ·3888824$

82

$5 \quad 89$

$\log \sin 15° 5' = \bar{1}·4153468$

$\frac{20}{60} \times 4684 = 1561$

$\bar{1}·4155029$

$\therefore a = 2·44845$.

Let h = the altitude; then $h = b \sin C$;

$$\therefore \log h = \log 2{\cdot}7402 + \log \sin 15^\circ\ 5'\ 20''$$
$$= {\cdot}4379823 + \bar{1}{\cdot}4155029$$
$$= \bar{1}{\cdot}8532852$$
$$\log {\cdot}71332 = \bar{1}{\cdot}8532844$$
$$\begin{array}{rr} & 8 \\ 1 & 6 \\ \hline \end{array}$$

$\therefore$ altitude $= {\cdot}713321$.

34. Let $b = 105{\cdot}25$, $c = 76{\cdot}75$, $B - C = 17^\circ\ 48''$;

then
$$\cot \frac{A}{2} = \frac{b+c}{b-c} \tan \frac{B-C}{2} = \frac{182}{28{\cdot}5} \tan 8^\circ\ 54'.$$

$$\log \cot \frac{A}{2} = \log 182 + \log \tan 8^\circ\ 54' - \log 28{\cdot}5$$
$$= 2{\cdot}2600714 + \bar{1}{\cdot}1947802 - 1{\cdot}4548449$$
$$= {\cdot}0000067 = \log \cot 45^\circ \text{ nearly.}$$
$$\therefore A = 90^\circ \text{ nearly.}$$

35. (1) $\sin C = \dfrac{c \sin A}{a} = \dfrac{36{\cdot}5 \sin 43^\circ\ 15'}{20}$;

$$\log \sin C = 1{\cdot}5622929 + \bar{1}{\cdot}8358066 - 1{\cdot}3010300$$
$$= {\cdot}0970695,$$

which is impossible, since $\sin C$ must be < 1.

(2) $\sin C = \dfrac{36{\cdot}5 \sin 43^\circ\ 15'}{30}$;

$$\log \sin 43^\circ\ 15' = \bar{1}{\cdot}8358066$$
$$\log 36{\cdot}5 = 1{\cdot}5622929$$
$$\overline{1{\cdot}3980995}$$
$$\log 30 = 1{\cdot}4771213$$
$$\log \sin C = \bar{1}{\cdot}9209782$$

Thus C is not a right angle, and since $a < c$ the solution is ambiguous.

(3) $\sin C = \dfrac{36{\cdot}5 \sin 43^\circ\ 15'}{45}$.

$$\log \sin C = 1{\cdot}3980995 - 1{\cdot}6532125$$
$$= \bar{1}{\cdot}7448870 \qquad \text{Diff. for } 60'' = 1890;$$
$$\log \sin 33^\circ\ 45' = \bar{1}{\cdot}7447390 \qquad \therefore \text{ prop}^{\text{l}}\text{. increase} = \frac{148}{189} \times 60'' = 47''.$$
$$\text{diff.} \quad 1480$$

$\therefore C = 33^\circ\ 45'\ 47''$; $\therefore B = 102^\circ\ 59'\ 13''$.

Now $b=\dfrac{a\sin B}{\sin A}=\dfrac{45\cos 12^\circ\,59'\,13''}{\sin 43^\circ\,15'}$;

$\log\cos 12^\circ\,59'=\bar{1}\cdot9887531$

Subtract $\dfrac{13}{60}\times 292=\quad 63$

$\bar{1}\cdot9887468$

$\log 45\quad=1\cdot6532125$

$\bar{1}\cdot6419593$

$\log\sin 43^\circ\,15'=\bar{1}\cdot8358066$

$\bar{1}\cdot8061527$

$\log 63\cdot996\quad=1\cdot8061528$

Thus $b=63\cdot996$.

36. For the first part of the Example, see Art. 197.

$$\tan\theta=\frac{2\sqrt{17\cdot32\times 13\cdot47}}{3\cdot85}\sin 23^\circ\,36'\,30'';$$

$\log\sin 23^\circ\,36'=\bar{1}\cdot6024388$

$\dfrac{30}{60}\times 2890=\quad 1445$

$\log 2\quad=\ \cdot3010300$

$1\cdot1839578$

$\bar{1}\cdot0875711$

$\log 3\cdot85\quad=\ \cdot5854607$

$\log\tan\theta\quad=\ \cdot5021104$

$\log\tan 72^\circ\,31'=\ \cdot5017184$

3920

$\therefore\ \theta=72^\circ\,31'\,53''$.

$\log 17\cdot32=1\cdot2385479$

$\log 13\cdot47=1\cdot1293676$

$2\,)\,2\cdot3679155$

$1\cdot1839578$

Diff. for $60''=4410$;

$\therefore$ prop^{l}. increase $=\dfrac{392}{441}\times 60''=53''$.

Again $c=(a-b)\sec\theta$;

$\log 3\cdot85\quad=\ \cdot5854607$

$\log\sec 72^\circ\,31'=\ \cdot5222591$

$\dfrac{53}{60}\times 4013=\quad 3545$

$1\cdot1080743$

$\log 12\cdot825\quad=1\cdot1080574$

169

$5\qquad 170$

$\therefore\ c=12\cdot8255$.

37. See Art. 195.

$$\tan\phi = \frac{44{\cdot}1}{10{\cdot}5}\tan 22^\circ\,36';$$

$\log 44{\cdot}1 = 1{\cdot}6444386$
$\log \tan 22^\circ\,36' = \bar{1}{\cdot}6193645$
$1{\cdot}2638031$
$\log 10{\cdot}5 = 1{\cdot}0211893$
$\log \tan\phi = \cdot 2426138$
$\log \tan 60^\circ\,13' = \cdot 2423617$
diff. 2521

Diff. for $60'' = 2930$;

$\therefore$ prop^{l}. increase $= \frac{2521}{2930} \times 60'' = 52''$.

$$\therefore \phi = 60^\circ\,13'\,52''.$$

Again $c = (a-b)\cos\frac{C}{2}\sec\phi$;

$\log 10{\cdot}5 = 1{\cdot}0211893$
$\log \cos 22^\circ\,36' = \bar{1}{\cdot}9653006$
$\log \sec 60^\circ\,13' = \cdot 3038870$

$\frac{52}{60} \times 2208 = 1914$

$\log c = \overline{1{\cdot}2905683}$
$\log 19{\cdot}523 = 1{\cdot}2905466$
217
$9 \quad 200$
170
$8 \quad 178$

$\therefore c = 19{\cdot}52398.$

EXAMPLES. XVII. a. PAGE 185.

1. See figure on page 184.

Let PC represent the cliff, and A and B the two objects. Then $PC = 200$ ft.; $\angle PAC = 30^\circ$, $\angle PBC = 45^\circ$.

$$AB = \frac{BP\sin APB}{\sin PAB} = \frac{BP\sin 15^\circ}{\sin 30^\circ};$$

and

$$BP = \frac{PC}{\sin PBC} = \frac{200}{\sin 45^\circ};$$

$$\therefore AB = \frac{200\sin 15}{\sin 45\sin 30} = 200\,(\sqrt{3}-1) = 146{\cdot}4 \text{ ft.}$$

2. See figure on page 185.

Let P represent the mountain top, and A, B the two positions of the observer.

Then $\angle PAC=15°$, $\angle PBC=75°$, $AB=1$ mile.

Let x be the height of mountain in feet;

then $x=PB\sin 75°$; and $PB=\dfrac{AB\sin 15°}{\sin 60°}$;

$$\therefore\ x=\frac{1760\,.\,3\,.\sin 15°\sin 75°}{\sin 60°}=\frac{880\,.\,3\,.(\cos 60°-\cos 90°)}{\sin 60°}$$

$$=\frac{880\,.\,3}{\sqrt{3}}=880\sqrt{3}=1524\text{ ft.}$$

3. Let A, B be the position of the two forts, P the first position of the ship, and Q its position after moving 4 miles towards A;

then $PQ=4$ miles, $\angle QPB=30°$, $\angle AQB=48°$; $\therefore\ \angle QBP=18°$.

$$\therefore QB=\frac{QP\sin 30°}{\sin 18°}=\frac{8}{\sqrt{5}-1}=2(\sqrt{5}+1)=6{\cdot}472\text{ miles.}$$

4. See figure on page 184.

Let PC represent the tower and A, B the two objects; then

$$PC=h,\ \angle PAC=45°-A,\ \angle PBC=45°+A,\ \angle APB=2A;$$

$$\therefore\ AB=\frac{PB\sin 2A}{\sin(45°-A)},\ \text{and}\ PB=\frac{h}{\sin(45°+A)};$$

$$\therefore\ AB=\frac{2h\sin 2A}{2\sin(45°-A)\sin(45°+A)}=\frac{2h\sin 2A}{\cos 2A-\cos 90°}=2h\tan 2A.$$

5. In the figure on page 184, take D in AB, so that $CD=CP$; then

$$AD=a\text{ feet},\ DB=b\text{ feet},\ PC=x\text{ feet, suppose.}$$

Since $\angle CDP=45°=\angle DPC$, $\therefore\ DC=PC=x$.

Also $AC=x+a$, $BC=x-b$, $\angle BPC=A$.

Now from $\triangle PAC$, $\tan A=\dfrac{x}{x+a}=\dfrac{x-b}{x}$, in $\triangle BPC$.

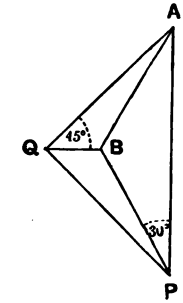
$$\therefore\ x^2=(x+a)(x-b);\ \text{whence}\ x=\frac{ab}{a-b}.$$

6. Let P, Q be the two positions of the observer.

Then from the figure, since

$$\angle AQP=90°,\ \angle QPA=45°;$$

$$\therefore\ QA=QP=1 \text{ mile.}$$



And $\triangle$'s ABQ, PBQ are equal in all respects (Euc. I. 4);

$$\therefore\ \angle ABQ=\angle PBQ=120°;$$

$$\therefore\ AB=\frac{AQ\sin 45°}{\sin 120°}=\frac{\sqrt{2}}{\sqrt{3}}=\frac{1}{3}\sqrt{6}=\cdot 816 \text{ miles.}$$

7. Let A be the base, and B the top of the tower, and let C be the point of observation 40 feet up the hill.

Then $AC=40$ ft., $\angle BAC=90°-9°=81°$, $\angle BCA=54°$; $\therefore\ \angle ABC=45°$;

$$\therefore\ AB=\frac{AC\sin 54°}{\sin 45°}=10\sqrt{2}\,(\sqrt{5}+1)=45\cdot 76 \text{ feet.}$$

8. See figure on page 186.

Let PA represent the tower, and PB the flagstaff.

Then $PB=c$ feet, $\angle PCA=\alpha$, $\angle PCB=\beta$, and we have

$$x=CP\sin\alpha=\frac{c\cos(\alpha+\beta)\sin\alpha}{\sin\beta}.$$

9. See figure on page 186.

Let BP represent the flagstaff, and PA the wall, C the point of observation.

Let $\angle BCP=\alpha$, $\angle PCA=\theta$, $CA=a$,

Then $\tan\alpha=\cdot 5$, $BP=20$ ft., $PA=10$ ft.

Now
$$\tan(\theta+\alpha)=\frac{30}{a},\quad \tan\theta=\frac{10}{a};$$

$$\therefore\ 3\tan\theta=\tan(\theta+\alpha)=\frac{2\tan\theta+1}{2-\tan\theta};$$

$$\therefore\ 3\tan^2\theta-4\tan\theta+1=0;\ \text{whence } \tan\theta=1 \text{ or } \frac{1}{3}.$$

10. See figure on page 186.

Let BP represent the statue, PA the tower, and C the point of observation.

Let $\angle PCA=\alpha$, $\angle BCP=\beta$, $BP=x$ feet.

Then $PA=25$ ft., $CA=60$ ft.;

$$\therefore \tan\alpha=\frac{25}{60}=\frac{5}{12}, \text{ and } \tan\beta=\cdot125=\frac{1}{8}.$$

Now $$x+25=60\tan(\alpha+\beta)=60\left(\frac{\frac{1}{8}+\frac{5}{12}}{1-\frac{5}{96}}\right)=34\tfrac{2}{7};$$

$\therefore$ height of statue $=9\frac{2}{7}$ feet.

11. See figure and example on page 187.

Here we have $BC=9$ ft., $BD=289$ ft., $BE=324$ ft.;

$$\therefore \tan(\alpha+\theta)=\frac{324}{x}; \quad \tan\alpha=\frac{289}{x}; \quad \tan\theta=\frac{9}{x}.$$

But $$\tan(\alpha+\theta)=\frac{\tan\alpha+\tan\theta}{1-\tan\alpha\tan\theta};$$

$$\therefore \frac{324}{x}=\frac{\frac{289}{x}+\frac{9}{x}}{1-\frac{289}{x}\cdot\frac{9}{x}}=\frac{298}{x}\cdot\frac{x^2}{x^2-289\times9};$$

$$\therefore 324x^2-324\times289\times9=298x^2;$$

$$\therefore 26x^2=324\times289\times9=18^2\times17^2\times3^2;$$

$$\therefore x^2=\frac{18^2\times51^2}{26}=18^2\times\left(\frac{50}{5}\right)^2 \text{ nearly};$$

thus $x=180$ ft. nearly.

12. See figure on page 187.

Let BD represent the column, DE the statue, BC the man standing by the column, A the point on the opposite bank of the river.

Then $BC=6$ ft., $BD=192$ ft., $BE=216$ ft.

Let $AB=x$ ft., $\angle EAD=\angle CAB=\theta$, $\angle DAB=\alpha$.

Then $$\tan(\alpha+\theta)=\frac{216}{x}; \ \tan\alpha=\frac{192}{x}; \ \tan\theta=\frac{6}{x};$$

$$\therefore \frac{216}{x}=\frac{\frac{192}{x}+\frac{6}{x}}{1-\frac{6}{x}\times\frac{192}{x}}=\frac{198}{x}\cdot\frac{x^2}{x^2-6\times192}.$$

From this equation we obtain $x=48\sqrt{6}$;

$\therefore$ breadth of river $=48\sqrt{6}=117{\cdot}6$ feet nearly.

13. We have at once from a figure,

$$\tan\alpha=\frac{a}{x}, \ \tan\beta=\frac{b}{x}, \ \tan\gamma=\frac{c}{x}.$$

Now $$\alpha+\beta+\gamma=180°;$$

$$\therefore \tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma;$$

$$\therefore \frac{a}{x}+\frac{b}{x}+\frac{c}{x}=\frac{abc}{x^3};$$

that is, $$(a+b+c)\,x^2=abc.$$

14. See figure on page 188.

Let P be the top of the hill, A and B the points of observation; then $$\angle PAC=45°,\ \angle BAC=15°,\ \angle PDC=75°;$$

$$\therefore \angle BPA=75°-45°=30°=\angle PAB;$$

$$\therefore PA=2AB\cos 30°=500\sqrt{3}\text{ yards};$$

$$\therefore \text{height of hill}=PC=PA\sin 45°=250\sqrt{6}\text{ yards}$$

$$=750\sqrt{6}\text{ feet.}$$

15. We have $\angle CBA=30°$, $\angle BCA=135°$; $\therefore \angle BAC=15°$;

$$\therefore AB=\frac{BC\sin 135°}{\sin 15°}=1760\times 3\times\frac{2}{\sqrt{3}-1}\text{ ft.}$$

$$=1760\times 3\,(\sqrt{3}+1)\text{ feet;}$$

$\therefore$ height of mountain $=AB\sin 60°=880\times 3\,(3+\sqrt{3})=12492$ ft.

16. See figure on page 188.

Let A, B be the two points of observation and P the top of the hill; then in the figure $\angle PAC=\alpha$, $\angle PDC=\gamma$, $AB=C$ ft.;

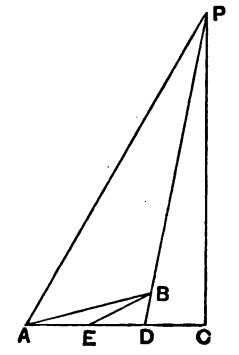
$$\therefore \angle APB=\gamma-\alpha,\ \angle ABP=\pi-(\gamma-\beta),$$

and $$AP=\frac{AB\sin(\gamma-\beta)}{\sin(\gamma-\alpha)};$$

$$\therefore \text{height of hill}=AP\sin\alpha$$

$$=c\sin\alpha\sin(\gamma-\beta)\operatorname{cosec}(\gamma-\alpha)\text{ feet.}$$

17. In the figure let P be the top of the mountain, and A, B the two points of observation.



Then $AE=EB=800$ ft.;

$$\angle BAE=15°,\ \angle BED=30°.$$

Also $$\angle PDC=75°,\ \angle PAC=60°;$$

$$\therefore \angle APD=15°.$$

From $\triangle ABE$ we have

$$AB=2AE\cos 15°=1600\cos 15°\text{ ft.;}$$

and from $\triangle APB$, $AP=\dfrac{AB\sin 120°}{\sin 15°}$

$$=800\sqrt{3}\cot 15°=800\sqrt{3}\,(2+\sqrt{3})\text{ ft.;}$$

$\therefore$ height of mountain $=AP\sin 60°$

$$=400\times 3\,(2+\sqrt{3})=4478\text{ ft.}$$

approximately.

EXAMPLES. XVII. b. PAGE 190.

1. In the figure of page 186 let $BP=25$ ft., $PA=15$ ft., $CA=x$ feet.

Then $$\frac{BC}{x}=\frac{25}{15}=\frac{5}{3} \quad \text{[Euc. VI. 3]};$$

$$\therefore \frac{x^2+40^2}{x^2}=\frac{25}{9}; \text{ whence } x=30;$$

thus the width of the road is 30 ft.

2. Let C be the position of the observer, B the top of the statue, A the foot of the column.

Then if $CB=x$ feet, we have

$$\frac{x}{CA}=\frac{a}{3a}=\frac{1}{3} \quad \text{[Euc. VI. 3]},$$

that is, $\dfrac{x}{\sqrt{x^2+16a^2}}=\dfrac{1}{3}$; whence $x=a\sqrt{2}$.

3. See figure on page 189.

Let BL be the flagstaff, LA the tower, C the observer;

then $BL=a,\ LA=b,\ DA=CE=d,\ EA=CD=h;$

also $BC^2=CE^2+EB^2=d^2+(a+b-h)^2,$

$CA^2=CE^2+EA^2=d^2+h^2.$

Now $$\frac{BC}{CA}=\frac{BL}{LA}; \quad \therefore \frac{(a+b-h)^2+d^2}{h^2+d^2}=\frac{a^2}{b^2};$$

$$\therefore \frac{(a+b)^2-2h(a+b)}{h^2+d^2}=\frac{a^2-b^2}{b^2};$$

or $$\frac{a+b-2h}{h^2+d^2}=\frac{a-b}{b^2};$$

whence $$(a-b)d^2=(a+b)b^2-2b^2h-(a-b)h^2.$$

4. See figure on page 190.

From O, the centre of the circle, draw OL, OM perpendicular to AB and DC respectively; then L, M bisect AB, DC. Let $DC=2x$ feet.

Then $$\angle\beta=\frac{1}{2}\angle DOC \text{ at centre}=\angle COM.$$

Now $$CD=2x=2CM=2OM\tan\beta$$
$$=2EL\tan\beta=(a+b)\tan\beta,$$

since $$2EL=EB+EA=a+b.$$

5. With the same figure and notation as in the last Example, we have $ED=AB=20$ ft., and $\beta=45°$.

$$\therefore 2x=(EB+EA)\tan 45°=2EB+20;$$
$$\therefore x=EB+10.$$

Again $$ED \cdot EC = EB \cdot EA = EB\,(EB+20).$$

$$\therefore\ 20\,(20+2x) = (x-10)\,(x+10);$$

whence $$x = 50.$$

Thus the height of the column is 100 ft.

6. Take the figure on page 190, interchanging the letters A and B. Then we have

$$\angle BDA = \angle BCA = \alpha - \beta;\quad \angle ABD = \angle ACE = 90^\circ - \alpha;$$

$$\therefore\ \angle DBC = \beta - \angle ABD = \alpha + \beta - 90^\circ.$$

Now from $\triangle CBD$,

$$CD = \frac{BD \sin DBC}{\sin BCE} = \frac{BD \sin(\alpha+\beta-90^\circ)}{\sin BCE};$$

$$BD = \frac{AB \sin BAD}{\sin BDA} = \frac{a \sin BCE}{\sin(\alpha-\beta)};$$

$$\therefore\ CD = a \sin(\alpha+\beta-90^\circ)\operatorname{cosec}(\alpha-\beta).$$

7. See figure on page 191.

Let CB be the pillar, BA the pedestal, E the point where the pillar subtends its maximum angle 30°.

Then using the same construction as in Ex. III. page 191, we have

$$a = 30^\circ,\ EA = 60 \text{ ft.}$$

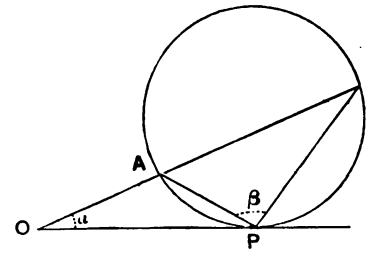
Now $$\angle AEB = \angle ECB = \frac{1}{2}\angle EDB = \frac{1}{2}(90^\circ - 30^\circ) = 30^\circ.$$

$$CB = 2CF = 2DF \tan 30^\circ = 2 \times 60 \times \frac{1}{\sqrt{3}} = 40\sqrt{3} \text{ ft.}$$

8. Let O, P be the two positions of the observer; let

$$\angle APO = \theta.$$

Then $\angle ABP$, in alternate segment, $= \theta$.



Now $$AB = \frac{AP \sin\beta}{\sin\theta}$$

$$= \frac{c \sin\alpha \sin\beta}{\sin\theta \sin(\alpha+\theta)}$$

$$= \frac{2c \sin\alpha \sin\beta}{\cos\alpha - \cos(\alpha+2\theta)}.$$

But, from $\triangle OPB$, $\alpha + 2\theta + \beta = 180^\circ$;

$$\therefore\ AB = 2c \sin\alpha \sin\beta / (\cos\alpha + \cos\beta).$$

9. Let OA be the tower, AB the flagstaff, P, Q the points at which the flagstaff subtends equal angles, R the point at which it subtends the greatest possible angle; then since $\angle APB = \angle AQB$;

$$\therefore\ B, A, P, Q \text{ are concyclic and } OP \cdot OQ = OA \cdot OB.$$

Again since ARB is the greatest angle subtended at a point in OQ by the str. line AB; $\therefore$ a circle can be drawn to pass through A, B and touch OQ at R;

$$\therefore\ OA \cdot OB = OR^2;$$

$$\therefore\ OP \cdot OQ = OR^2;$$

that is OP, OR, OQ are in geometrical progression.

10. Here $ABCD$ is a cyclic quadrilateral;

$$\therefore\ \frac{AB}{\sin ADB} = \frac{BD}{\sin BAD} = \frac{BD}{\sin BCD} = \frac{DC}{\sin CBD};$$

$$\therefore\ AB \sin CBD = CD \sin ADB.$$

11. Since $ABED$ is a cyclic quadrilateral, we have

$$\angle ADC = \angle EBC = \gamma, \text{ and } \angle BDC = \beta.$$

Also $$\angle BDA = \gamma - \beta, \text{ and } \angle ACE = \pi - (\alpha + \beta + \gamma).$$

$$\therefore\ BC = \frac{BD \sin \beta}{\sin (\alpha + \beta + \gamma)} = \frac{AB \sin (\alpha + \beta) \sin \beta}{\sin (\gamma - \beta) \sin (\alpha + \beta + \gamma)}.$$

12. Here the points P, Q, R, S, A are concyclic, and

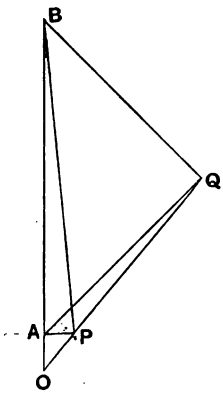
$$\angle RAS = \angle PAQ,$$

since AR, AS are perpendicular to AP, AQ;

$$\therefore\ PQ = RS = \sqrt{400 + 100 - 2 \times 200 \cos 30^\circ}$$

$$= \sqrt{500 - 200\sqrt{3}} = 12{\cdot}4 \text{ ft. nearly.}$$

13. Let A, B be the two beacons, P, Q the positions of the ship at the end of 3 min. and 21 min. respectively. Let $\angle ABP = \alpha$, $\angle PBQ = \theta$. Then it is easily seen that



$$\angle OAP = 90^\circ + \alpha,$$

$$\angle OAQ = 90^\circ + \alpha + \theta.$$

Also from the $\triangle OBQ$ we have

$$\alpha + \theta + 90^\circ + \alpha = 135^\circ,$$

so that $$\alpha + \theta = 45^\circ - \alpha.$$

If $$PO = x, \quad OQ = 7x;$$

hence $$x = \frac{AP}{\sin 45^\circ} \sin (90^\circ + \alpha)$$

$$= AB\sqrt{2} \cdot \sin \alpha \cos \alpha$$

$$= \frac{5}{\sqrt{2}} \sin 2\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again $$7x = \frac{AQ}{\sin 45^\circ} \sin (90^\circ + \alpha + \theta)$$

$$= AB\sqrt{2} \sin (\alpha + \theta) \cos (\alpha + \theta)$$

$$= \frac{5}{\sqrt{2}} \sin 2(\alpha + \theta) = \frac{5}{\sqrt{2}} \sin (90 - 2\alpha) = \frac{5}{\sqrt{2}} \cos 2\alpha \ldots\ldots\ldots\ldots (2).$$

Squaring and adding (1) and (2), we have

$$50x^2=\frac{25}{2};\ \therefore\ x=\frac{1}{2}\text{ mile.}$$

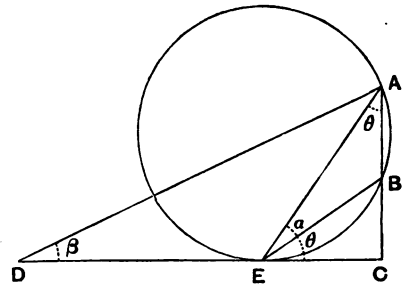
$\therefore$ the ship sails a mile in 6 min., or at the rate of 10 miles an hour.

Again if y miles be the distance from O at which the beacons subtend the greatest angle, we have

$$y^2=OP\,.\,OQ=\frac{7}{4};\ \therefore\ y=\frac{\sqrt{7}}{2}.$$

And the ship will travel this distance in $\frac{\sqrt{7}}{2}\times\frac{60}{10}$, or $3\sqrt{7}$ minutes.

14. Let AB be the flagstaff, BC the tower, D and E the first and second positions of the observer respectively.



Then since AB subtends the maximum angle at E, the circle round ABE touches DC at E, so that

$\angle BEC=\angle EAB=\theta$, suppose.

$\therefore\ \alpha+\theta=\angle EBC=90^\circ-\theta;$

$\therefore\ 2\theta=90^\circ-\alpha;$

also $\angle EAD=\alpha+\theta-\beta$

$=90^\circ-\theta-\beta.$

Now $$AB=\frac{AE\sin\alpha}{\sin(\theta+\alpha)},\quad AE=\frac{ED\sin\beta}{\sin(90^\circ-\theta-\beta)}=\frac{a\sin\beta}{\cos(\theta+\beta)};$$

$$\therefore\ AB=\frac{a\sin\alpha\sin\beta}{\sin(\theta+\alpha)\cos(\theta+\beta)}=\frac{2a\sin\alpha\sin\beta}{\sin(2\theta+\alpha+\beta)+\sin(\alpha-\beta)}$$

$$=\frac{2a\sin\alpha\sin\beta}{\cos\beta+\sin(\alpha-\beta)}.$$

EXAMPLES. XVII. c. PAGE 195.

1. Let A be the top of the hill and B its projection on the horizontal plane through P, Q.

Let $AB=x$ yards;

then $BP=BA=x,$

$BQ=BA\cot 30^\circ=x\sqrt{3};$

$\therefore\ 3x^2=x^2+500^2;$

$\therefore\ x=250\sqrt{2};$

at is, height of the hill $=250\sqrt{2}$ yards $=1060{\cdot}5$ feet.

2. Let P be the top, and Q the bottom of the spire;

then $$AQ = 250 \cot 60° = \frac{250}{\sqrt{3}} \text{ feet},$$

$$BQ = 250 \cot 30° = 250\sqrt{3} \text{ feet};$$

$$\therefore AB^2 = BQ^2 - AQ^2 = 250^2 \left(3 - \frac{1}{3}\right);$$

$$\therefore AB = 250 \cdot \frac{2\sqrt{2}}{\sqrt{3}} = \frac{500\sqrt{6}}{3} \text{ feet}.$$

3. Let P be the top and Q the bottom of the tower;

then $\angle PAQ = 60°$; $\therefore QA = 360 \cot 60° = \frac{360}{\sqrt{3}}$ feet and $QB = QP = 360$ feet;

$$\therefore \text{ breadth of river} = AB = \sqrt{BQ^2 - QA^2} = 360\sqrt{1 - \frac{1}{3}} = 120\sqrt{6} \text{ feet}.$$

4. See figure on page 194.

Let CD be the steeple, then $\angle CAD = 45°$; $\therefore CA = CD = x$;

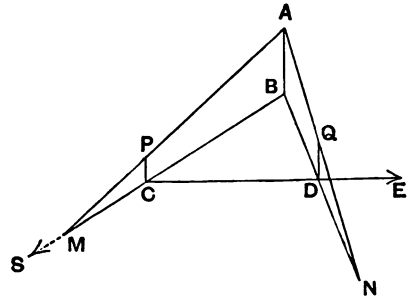
and $\angle CBD = 15°$; $\therefore CB = x \cot 15° = x(2 + \sqrt{3})$;

$$\therefore x^2(7 + 4\sqrt{3}) = x^2 + 4a^2;$$

$$\therefore x^2 = \frac{2a^2}{2\sqrt{3} + 3} = \frac{a^2(4 - 2\sqrt{3})}{\sqrt{3}};$$

$$\therefore \text{ height of steeple} = \frac{a(\sqrt{3} - 1)}{\sqrt[4]{3}} = a(3^{\frac{1}{4}} - 3^{-\frac{1}{4}}).$$

5. Let AB be the lighthouse, and let CP, DQ represent the two positions of the observer, M, N the extremities of his shadow at each place.



Let $AB = x$ feet, then $x : BM = PC : CM = 6 : 24$;

$$\therefore BC = BM - CM = 4x - 24.$$

Also $$x : BN = QD : DN = 6 : 30;$$

$$\therefore BD = BN - DN = 5x - 30;$$

$$\therefore 25(x-6)^2 = 300^2 + 16(x-6)^2;$$

$$\therefore 9(x-6)^2 = 300^2;$$

whence $$x = 106 \text{ or } -94;$$

$\therefore$ the light is 106 feet from the ground.

6. Let A be the balloon, and C, B, D the points of observation at which the angles of elevation of the balloon are 60°, 30°, 45° respectively.

Let x yards be the height of the balloon;

then $AB = x \operatorname{cosec} 30° = 2x$, $AC = x \operatorname{cosec} 60° = \frac{2x}{\sqrt{3}}$, $AD = x \operatorname{cosec} 45° = x\sqrt{2}$.

Now $$2AD^2 + 2BD^2 = AB^2 + AC^2;$$

$$\therefore 4x^2 + 2 \times 880^2 = 4x^2 + \frac{4x^2}{3};$$

whence $$x = 880 \cdot \sqrt{\frac{3}{2}} = 440\sqrt{6};$$

$$\therefore \text{height of balloon} = 440\sqrt{6} \text{ yards.}$$

7. Let A be the top of the mountain, BC the base of length $2a$, D its middle point.

Let x be the height of the mountain;

then $$AB = AC = x \operatorname{cosec} \theta,$$

$$AD = x \operatorname{cosec} \phi.$$

And we have $$AD^2 + BD^2 = AB^2;$$

$$\therefore x^2 \operatorname{cosec}^2 \phi + a^2 = x^2 \operatorname{cosec}^2 \theta;$$

$$\therefore x^2 = \frac{a^2}{\operatorname{cosec}^2 \theta - \operatorname{cosec}^2 \phi} = \frac{a^2 \sin^2\theta \cos^2\theta}{\sin^2\phi - \sin^2\theta},$$

$$\therefore x = \frac{a \sin\theta \cos\theta}{\sqrt{\sin(\phi+\theta)\sin(\phi-\theta)}};$$

$\therefore$ height of mountain is $a \sin\theta \cos\theta \sqrt{\operatorname{cosec}(\phi+\theta)\operatorname{cosec}(\phi-\theta)}$.

8. Let AB, CD be the two vertical poles, E the point in the line BD joining their feet at which each subtends an angle α, and F any point in the horizontal plane such that $\angle DFB$ is a right angle;

then $$\angle CED = \angle AEB = \alpha, \quad \angle AFB = \beta, \quad \angle CED = \gamma;$$

and $$EB = AB \cot AEB = a \cot\alpha, \quad ED = b \cot\alpha;$$

$$\therefore BD = (a+b)\cot\alpha;$$

$$BF = AB \cot AFB = a\cot\beta, \quad DF = b\cot\gamma;$$

and $$BD^2 = BF^2 + DF^2;$$

$$\therefore (a+b)^2 \cot^2\alpha = a^2\cot^2\beta + b^2\cot^2\gamma.$$

9. Let A be the top of the hill, B its projection on the horizontal plane in which the road lies, C, D, E the three consecutive milestones whose angles of depression are observed.

Then $\angle ACB=\alpha,\ \angle ADB=\beta,\ \angle AEB=\gamma,$

and $ED=DC=1760\times 3$ feet.

Let x feet be the height of the hill;

then since $BC^2+BE^2=2BD^2+2DE^2;$

$$\therefore\ x^2\cot^2\alpha+x^2\cot^2\gamma=2x^2\cot^2\beta+2\times 1760^2\times 3^2;$$

$$\therefore\ x=\frac{5280\sqrt{2}}{\sqrt{\cot^2\alpha-2\cot^2\beta+\cot^2\gamma}}.$$

10. Let P, Q be the two positions of the observer.

Let x feet be the height of the chimneys;

then $$AP=x\cot 60^\circ=\frac{x}{\sqrt{3}},\ AQ=AB=x;$$

$$\therefore\ x^2=\frac{x^2}{3}+80^2;$$

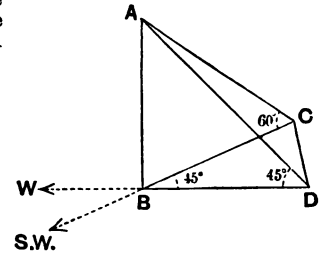
whence $x=40\sqrt{6}$; $\therefore$ height of chimney $=40\sqrt{6}$ feet.

Again $CQ=x\cot 30^\circ=x\sqrt{3};$

$$\therefore\ CP=\sqrt{CQ^2-PQ^2}=\sqrt{3\times 40^2\times 6-80^2}=40\sqrt{14};$$

$$\therefore\ AC=40\sqrt{14}+40\sqrt{2}=40\,(\sqrt{14}+\sqrt{2})=206 \text{ feet nearly.}$$

11. Let A be the balloon, C, D the positions of the two observers, B the point vertically below A in the horizontal plane of the observers.



Let $AB=x$ yds. = height of the balloon;

then $CD=500$ yards,

$$BC=x\cot 60^\circ=\frac{x}{\sqrt{3}},\ BD=x;$$

and $CD^2=BC^2+BD^2-2BC\,.\,BD\cos 45^\circ;$

$$\therefore\ 500^2=\frac{x^2}{3}+x^2-\frac{x^2\sqrt{6}}{3};$$

whence $$x^2=50^2\,(120+30\sqrt{6});$$

$$\therefore \text{ height of balloon}=50\sqrt{120+30\sqrt{6}}=696 \text{ yds. nearly.}$$

12. In the diagram on page 195, let AC be a line of greatest slope on the hill, and let CH be the railway.

Then $\angle ACB=\alpha$, $\angle BCG=x$, $\angle HCG=\beta$, and $\angle CBG$ is a right angle. Let $AB=HG=h$.

Then $$BC=h\cot\alpha,\ CG=h\cot\beta.$$

$$\therefore\ \cos x=\frac{BC}{CG}=\cot\alpha\tan\beta.$$

EXAMPLES. XVII. d. PAGE 197.

1. Let C_1, C_2 be the positions of the two churches, M the position of the man. Then MC_1C_2 is an isosceles triangle and $\angle MC_1C_2 = 84° 17' 20''$.

If x miles be the height of the balloon, we have $x = \frac{1}{2}\tan 84° 17' 20''$.

$$\begin{aligned}
\log \tan 84° 17' &= \cdot 9995353 \\
\tfrac{20}{60} \times 12764 &= \quad 4255 \\
&\overline{\cdot 9999608} \\
\log 2 &= \cdot 3010300 \\
\log x &= \overline{\cdot 6989308} \\
\log 4 \cdot 9995 &= \cdot 6989266
\end{aligned}$$

$\therefore$ the height of the balloon is 5 miles nearly.

2. Let DC be the pole, then from a figure we have

$$DC = BD \sin 10° 45' = \frac{100 \sin 5° 30' \times \sin 10° 45'}{\sin 5° 15'}.$$

$$\begin{aligned}
\log 100 + \log \sin 5° 30' &= \cdot 9815729 \\
\log \sin 10° 45' &= \bar{1} \cdot 2707348 \\
&\overline{\cdot 2523077} \\
\log \sin 5° 15' &= \bar{2} \cdot 9614288 \\
\log DC &= \bar{1} \cdot 2908789 \\
\log 19 \cdot 537 &= 1 \cdot 2908579 \\
&\overline{210} \\
9 \quad &200 \\
&\overline{100} \\
5 \quad &111
\end{aligned}$$

$\therefore DC = 19 \cdot 53795$ yards.

Again $BC = CD \cot 10° 45'$.

$$\begin{aligned}
\log CD &= 1 \cdot 2908789 \\
\log \cot 10° 45' &= \cdot 7215758 \\
\log BC &= \bar{2} \cdot 0124547 \\
\log 102 \cdot 90 &= 2 \cdot 0124154 \\
&\overline{393} \\
9 \quad &379 \\
&\overline{140} \\
3 \quad &126
\end{aligned}$$

$\therefore BC = 102 \cdot 9093$ yards.

3. If x feet be the required height, we have

$$x = \frac{500 \sin 26° 33' 55'' \times \sin 12° 31' 46''}{\sin 14° 2' 9''}.$$

$$\begin{aligned}
\log \sin 26° 33' &= \bar{1} \cdot 6502868 \\
\tfrac{55}{60} \times 2527 \quad &2316 \\
\log \sin 12° 31' &= \bar{1} \cdot 3359062 \\
\tfrac{46}{60} \times 5687 \quad &4360 \\
\log 500 &= 2 \cdot 6989700 \\
&\overline{1 \cdot 6858306} \\
&\overline{\bar{1} \cdot 3847630} \\
\log x &= \overline{2 \cdot 3010676} \\
\log 200 \cdot 01 &= 2 \cdot 3010517 \\
&\overline{159} \\
7 \quad &152
\end{aligned}$$

$\therefore$ *height* $= 200 \cdot 017$ feet.

$$\begin{aligned}
\log \sin 14° 2' &= \bar{1} \cdot 3846873 \\
\tfrac{9}{60} \times 5051 \quad &757 \\
&\overline{\bar{1} \cdot 3847630}
\end{aligned}$$

4. Let B be the position of the boat, FC the flagstaff, CD the cliff, then, if $CD = x$ feet, we have

$$x = \frac{30 \sin 44° 13' \times \sin 43° 48'}{\sin 1° 59'}.$$

$\log 30 = 1{\cdot}4771213$
$\log \sin 44° 13' = \bar{1}{\cdot}8434655$
$\log \sin 43° 48' = \bar{1}{\cdot}8401959$
$\overline{1{\cdot}1607827}$
$\log \sin 1° 59' = \bar{2}{\cdot}5391863$
$\log x = 2{\cdot}6215964$
$\log 418{\cdot}40 = 2{\cdot}6215917$
47
$4 \quad 42$
50
$5 \quad 52$

$\therefore$ height $= 418{\cdot}4045$ ft.

Also $BD = x \cot 44° 13'$.

$\log x = 2{\cdot}6215964$
$\log \cot 44° 13' = \cdot 0118766$
$2{\cdot}6334730$
$\log 430{\cdot}00 = 2{\cdot}6334685$

$\therefore$ distance $= 430$ ft. nearly.

5. With the figure on page 184 we have

$AB = 5280$ ft., $\angle PAC = 5°$, $\angle PBC = 10°$, $\angle APB = 5°$, $BP = AB$, $PC = PB \sin 10°$.

$\therefore \log PC = \log 5280 + \log \sin 10°$
$= 3{\cdot}7226340 + \bar{1}{\cdot}2396702$
$= 2{\cdot}9623042$
$\log 916{\cdot}86 = 2{\cdot}9623030$
12
$2 \quad 10$
20
$4 \quad 19$

$\therefore PC = 916{\cdot}8624$ ft.

Again $BC = BP \cos 10°$
$= 1 \times \cos 10°$ miles
$= {\cdot}9848078$ miles.

6. Let B be the point in the road which is vertically below the observer A. Let DC be the telegraph post, and let a horizontal line through A meet DC in E. Then $\angle DAE = 17° 18' 35''$, $\angle ACB = 8° 32' 15''$.

$CB = 15 \cot 8° 32' 15''$.

$\log 15 = 1{\cdot}1760913$
$\log \cot 8° 32' = \ \cdot 8237761$
$1{\cdot}9998674$
subtract $\frac{15}{60} \times 8601 \quad 2150$
$\log CB = \overline{1{\cdot}9996524}$
$\log 99{\cdot}92 = \overline{1{\cdot}9996524}$

$\therefore CB = 99{\cdot}92$ ft.

$DE = EA \tan 17° 18' 35''$
$= CB \tan 17° 18' 35''$.

$\log CB = 1{\cdot}9996524$
$\log \tan 17° 18' = \bar{1}{\cdot}4934097$
$\frac{35}{60} \times 4448 = \quad 2595$
$\log DE = \overline{1{\cdot}4933216}$
$\log 31{\cdot}140 = \overline{1{\cdot}4933186}$

	30
2	28
	20
1	14

$\therefore DE = 31{\cdot}14021$.

Thus $CD = 46{\cdot}14021$ ft.

7. Here we may take the third figure on page 131. Then $AC = 60$ miles, $CB_2 = CB_1 = 30$ miles, $\angle CAB_2 = 20° 16'$.

$$\sin AB_1C = \frac{60}{30} \sin CAB_1 = 2 \sin 20° 16'.$$

$\log 2 = \ \cdot 3010300$
$\log \sin 20° 16' = \bar{1}{\cdot}5395653$
$\bar{1}{\cdot}8405953$
$\log \sin 43° 51' = \bar{1}{\cdot}8405908$
45

$\frac{45}{1315} \times 60'' = 2''$; $\therefore AB_1C = 43° 51' 2''$.

$\therefore \angle ACB_2 = 23° 35' 2''$, $\angle ACB_1 = 180° - 64° 7' 2''$.

Now $AB_2 = \dfrac{30 \sin 23° 35' 2''}{\sin 20° 16'}$.

$\log \sin 23° 35' = \bar{1}{\cdot}6021495$
$\frac{2}{60} \times 2893 = \quad 96$
$\log 30 = 1{\cdot}4771213$
$\bar{1}{\cdot}0792804$
$\log \sin 20° 16' = \bar{1}{\cdot}5395653$
$\log AB_2 = \overline{1{\cdot}5397151}$
$\log 34{\cdot}050 = 1{\cdot}5397032$
119

9	113
	60
5	63

$\therefore AB_2 = 34{\cdot}05095$ miles.

$AB_1 = \dfrac{30 \sin 64° 7' 2''}{\sin 20° 16'}$.

$\log \sin 64° 7' = \bar{1}{\cdot}9540904$
$\frac{2}{60} \times 613 = \quad 20$
$\log 30 = 1{\cdot}4771213$
$\bar{1}{\cdot}4312137$
$\log \sin 20° 16' = \bar{1}{\cdot}5395653$
$\log AB_1 = \overline{1{\cdot}8916484}$
$\log 77{\cdot}919 = 1{\cdot}8916434$
50

9	50

$\therefore AB_1 = 77{\cdot}9199$ miles.

Thus the train must travel at the rate of 11·550316 miles or 25·9733 miles per hour.

8. Let C be the doorstep, E the point of observation on the roof; then $CE=h$. Let AB be the spire and let ED drawn horizontally meet AB in D; then $\angle ACB=5a$, $\angle AED=4a$. Also $\angle EAC=a$. Let $AB=x$.

Then
$$x=AC\sin 5a=\frac{h\sin(90^\circ+4a)}{\sin a}.\sin 5a$$
$$=h\operatorname{cosec}a\cos 4a\sin 5a.$$
$$CB=x\cot 5a=h\operatorname{cosec}a\cos 4a\cos 5a.$$

In the particular case
$$a=7^\circ 17'39'',\ 4a=29^\circ 10'36'',\ 5a=36^\circ 28'15''.$$

$\log 39=1{\cdot}5910646$
$\log\sin 36^\circ 28'=\bar{1}{\cdot}7740459$
$\frac{15}{60}\times 1709=\quad 427$
$\log\operatorname{cosec}7^\circ 17'39''=\ {\cdot}8963210$
$\log\cos\ 29^\circ 10'36''=\bar{1}{\cdot}9410743$
$\log x=2{\cdot}2025485$
$\log 159{\cdot}42=2{\cdot}2025428$

	57
2	55
	20
1	27

$\log\operatorname{cosec}7^\circ 17'=\ {\cdot}8969627$
subtract $\frac{39}{60}\times 9873$ 6417
${\cdot}8963210$

$\log\cos 29^\circ 10'=\bar{1}{\cdot}9411166$
subtract $\frac{36}{60}\times 705$ 423
$\bar{1}{\cdot}9410743$

Thus the height $=159{\cdot}4221$ feet.

Again
$\log 39=1{\cdot}5910646$
$\log\operatorname{cosec}7^\circ 17'39''=\ {\cdot}8963210$
$\log\cos 29^\circ 10'36''=\bar{1}{\cdot}9410743$
$\log\cos 36^\circ 28'=\bar{1}{\cdot}9053656$
$2{\cdot}3338255$
subtract $\frac{15}{60}\times 934\quad 234$
$\log CB=2{\cdot}3338021$
$\log 215{\cdot}67=2{\cdot}3337897$

	124
6	121
	30
2	40

Thus the distance $=215{\cdot}6762$ feet.

EXAMPLES. XVIII. a. PAGE 206.

1. $\text{Area} = \frac{1}{2} \times 300 \times 120 \sin 150° = 300 \times 60 \times \frac{1}{2} = 9000$ sq. feet.

2. $2s = 171 + 204 + 195 = 570$;

$\therefore$ area $= \sqrt{285 \times 114 \times 81 \times 90} = \sqrt{10^2 \times 57^2 \times 27^2} = 15390$.

3. Let $a = 70$, $b = 147$, $c = 119$; then $s = 168$.

$\therefore \sin B = \frac{2}{70 \times 119} \sqrt{168 \times 98 \times 21 \times 49} = \frac{2 \times 12 \times 7 \times 49}{70 \times 119} = \frac{84}{85}$.

4. Let $a = 39$, $b = 40$, $c = 25$, and denote the perpendiculars by p_1, p_2, p_3.

Then area $= \sqrt{52 \times 13 \times 12 \times 27} = 12 \times 13 \times 3$.

$$\therefore p_1 = \frac{2\Delta}{a} = 24, \quad p_2 = \frac{2\Delta}{b} = \frac{117}{5}, \quad p_3 = \frac{2\Delta}{c} = \frac{936}{25}.$$

5. $\text{Area} = \frac{30^2 \sin 22\frac{1}{2}° \sin 112\frac{1}{2}°}{2 \sin 135°} = \frac{30^2 \sin 22\frac{1}{2}° \cos 22\frac{1}{2}°}{2 \sin 45°} = \frac{30^2}{4} = 225$ sq. ft.

6. The diagonal bisects the parallelogram;

$\therefore$ area $= 42 \times 32 \sin 30° = 672$ sq. feet.

7. Let a yds. be the length of a side.

Then $$\text{area} = a^2 \sin 150° = \frac{a^2}{2};$$

$$\therefore \frac{a^2}{2} = 648; \quad \therefore a = 36 \text{ yards.}$$

$\therefore$ length of a side is 36 yds.

8. $13 + 14 + 15 = 42$;

$$\therefore \Delta = \sqrt{21 \times 8 \times 7 \times 6} = 4 \times 3 \times 7.$$

$$\therefore R = \frac{13 \times 14 \times 15}{4\Delta} = \frac{65}{8} = 8\frac{1}{8}.$$

$$r = \frac{\Delta}{21} = 4.$$

9. $17 + 10 + 21 = 48$;

$$\therefore \Delta = \sqrt{24 \times 7 \times 14 \times 3} = 4 \times 3 \times 7.$$

$$\therefore r_1 = \frac{\Delta}{7} = 12, \quad r_2 = \frac{\Delta}{14} = 6, \quad r_3 = \frac{\Delta}{3} = 28.$$

10. $s - a = \frac{\Delta}{r_1} = 12$; $s - b = \frac{\Delta}{r_2} = 8$; $s - c = \frac{\Delta}{r_3} = 4$; $\therefore s = 24$;

$$\therefore a = 12, \ b = 16, \ c = 20.$$

11. $\sqrt{rr_1r_2r_3}=\sqrt{\frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}}=\Delta.$

12. $s(s-a)\tan\frac{A}{2}=s(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{s(s-a)(s-b)(s-c)}=\Delta.$

13. $$rr_1\cot\frac{A}{2}=\frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\frac{s(s-a)}{\Delta}=\Delta.$$

14. $$4Rrs=\frac{abc}{\Delta}\cdot\frac{\Delta}{s}\cdot s=abc.$$

15. $$r_1r_2r_3=\frac{\Delta^3}{(s-a)(s-b)(s-c)}=\frac{\Delta^3 s}{\Delta^2}=\Delta s=rs^2.$$

16. $$r\cot\frac{B}{2}\cot\frac{C}{2}=\frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}\cdot\cot\frac{B}{2}\cot\frac{C}{2}=\frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}=r_1.$$

17. $$\text{First side}=r\left(\frac{a}{2}+\frac{b}{2}+\frac{c}{2}\right)=rs=\Delta.$$

18. $$r_1r_2+rr_3=\frac{\Delta^2}{(s-a)(s-b)}+\frac{\Delta^2}{s(s-c)}=\frac{\Delta^2\{s(s-c)+(s-a)(s-b)\}}{\Delta^2}$$
$$=2s^2-(a+b+c)s+ab=ab.$$

20. $$r_1+r_2=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$
$$=4R\cos\frac{C}{2}\cdot\sin\frac{A+B}{2}=4R\cos^2\frac{C}{2}$$
$$=\frac{2c}{\sin C}\cdot\cos^2\frac{C}{2}=c\cot\frac{C}{2}.$$

21. As in Ex. 20, $r_1-r=4R\sin\frac{A}{2}\cdot\cos\frac{B+C}{2}=4R\sin^2\frac{A}{2}$,
$$r_2+r_3=4R\cos^2\frac{A}{2}.$$
$$\therefore\ (r_1-r)(r_2+r_3)=4R^2\sin^2A=a^2.$$

22. By the formulæ of Art. 212, we have
$$\frac{r_1}{r}=\cot\frac{B}{2}\cot\frac{C}{2};\ \therefore\ r_1\cot\frac{A}{2}=r\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

23. $$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s-a+s-b+s-c}{\Delta}=\frac{3s-2s}{\Delta}=\frac{s}{\Delta}=\frac{1}{r}.$$

24. $r_2r_3+r_3r_1+r_1r_2=\dfrac{\Delta^2}{(s-b)(s-c)}+\ldots+\ldots=s(s-a)+\ldots+\ldots$

$=3s^2-(a+b+c)s=s^2.$

25. As in Ex. 21, $r_2+r_3=4R\cos^2\dfrac{A}{2}$, $r_1-r=4R\sin^2\dfrac{A}{2}$.

$\therefore\ r_1+r_2+r_3-r=4R\left(\cos^2\dfrac{A}{2}+\sin^2\dfrac{A}{2}\right)=4R.$

26. $r_1+r_2=4R\cos^2\dfrac{C}{2}$, $r_3-r=4R\sin^2\dfrac{C}{2}$. [Ex. 21.]

$\therefore\ r+r_1+r_2-r_3=4R\left(\cos^2\dfrac{C}{2}-\sin^2\dfrac{C}{2}\right)=4R\cos C.$

27. $b^2\sin 2C+c^2\sin 2B=2b\sin C\,.\,b\cos C+2c\sin B\,.\,c\cos B$

$=2b\sin C\,(b\cos C+c\cos B)=2ab\sin C=4\Delta.$

28. $(a+b)\sec\dfrac{A-B}{2}=2R\,(\sin A+\sin B)\sec\dfrac{A-B}{2}$

$=4R\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}\sec\dfrac{A-B}{2}=4R\cos\dfrac{C}{2}.$

29. $a^2-b^2=4R^2(\sin^2A-\sin^2B)=4R^2\sin(A+B)\sin(A-B)$

$=4R^2\sin C\sin(A-B)=2Rc\sin(A-B).$

30. First side $=4R^2.\dfrac{\sin^2A-\sin^2B}{2}\cdot\dfrac{\sin A\sin B}{\sin(A-B)}$

$=\dfrac{4R^2\sin(A+B)\sin A\sin B}{2}$

$=\dfrac{2R\sin A\,.\,2R\sin B\,.\,\sin C}{2}=\dfrac{ab}{2}\sin C=\Delta.$

31. (1) We have $2\Delta=ap_1=bp_2=cp_3$;

$\therefore\ \dfrac{1}{p_1}+\dfrac{1}{p_2}+\dfrac{1}{p_3}=\dfrac{a+b+c}{2\Delta}=\dfrac{s}{\Delta}=\dfrac{1}{r}.$

(2) $\dfrac{1}{p_1}+\dfrac{1}{p_2}-\dfrac{1}{p_3}=\dfrac{a+b-c}{2\Delta}=\dfrac{s-c}{\Delta}=\dfrac{1}{r_3}.$

32. $(r_1-r)(r_2-r)(r_3-r)=64R^3\sin^2\dfrac{A}{2}\sin^2\dfrac{B}{2}\sin^2\dfrac{C}{2}$ [Ex. 21]

$=4Rr^2.$ [Art. 212.]

33. $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{(r_1-r)(r_2-r)(r_3-r)}{r^3r_1r_2r_3}=\frac{4Rr^2}{r^2\Delta^2}$ [Ex. 11]

$$=\frac{4R}{r^2s^2}.$$

34. $4\Delta(\cot A+\cot B+\cot C)=2bc\cos A+2ca\cos B+2ab\cos C$

$$=b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2$$

$$=a^2+b^2+c^2.$$

35. $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=\frac{(b-c)(s-a)+(c-a)(s-b)+(a-b)(s-c)}{\Delta}$

$$=\frac{s(b-c+c-a+a-b)-\{a(b-c)+b(c-a)+c(a-b)\}}{\Delta}=0.$$

36. $a^2b^2c^2(\sin 2A+\sin 2B+\sin 2C)=4a^2b^2c^2\sin A\sin B\sin C$

$$=4\,.\,bc\sin A\,.\,ca\sin B\,.\,ab\sin C$$

$$=32\Delta^3.$$

37. $a\cos A+b\cos B+c\cos C=2R(\sin A\cos A+\sin B\cos B+\sin C\cos C)$

$$=R(\sin 2A+\sin 2B+\sin 2C)$$

$$=4R\sin A\sin B\sin C.$$

38. $a\cot A+b\cot B+c\cot C=2R(\cos A+\cos B+\cos C)$

$$=2R\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=2(R+r).$$

39. $(b+c)\tan\frac{A}{2}=2R(\sin B+\sin C)\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=4R\cos\frac{B-C}{2}\cdot\cos\frac{B+C}{2}$

$$=2R(\cos B+\cos C);$$

$\therefore\ (b+c)\tan\frac{A}{2}+\text{two similar terms}=4R(\cos A+\cos B+\cos C).$

40. $r(\sin A+\sin B+\sin C)=4r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$=16R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$=2R\sin A\sin B\sin C.$$

41. $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=\frac{1}{2}(3+\cos A+\cos B+\cos C)$

$$=\frac{1}{2}\left(4+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=2+\frac{r}{2R}.$$

EXAMPLES. XVIII. b. PAGE 210.

1. By Art. 214, the area $=\frac{1}{2}.10.9.\sin 36°$

$=45\times\cdot588=26\cdot46$ sq. ft.

2. By Art. 215, the perimeter $=2.15.\frac{3}{2}.\tan 12°$

$=45\times\cdot213=9\cdot585$ yds.

The area $=15.\frac{9}{4}.\tan 12°=\frac{135\times\cdot213}{4}=7\cdot18875$ sq. yds.

3. In fig. of Art. 215 let $AB=2x=$ side of regular hexagon. Then

$$OD=x\cot 30°=x\sqrt{3}.$$

$\therefore$ area of inscribed circle $=3\pi x^2$.

Again, in fig. of Art. 214, if AB be the side of the hexagon,

$$OA=AD\operatorname{cosec}30°=2x.$$

$\therefore$ area of circumscribed circle $=4\pi x^2$.

$\therefore$ ratio of areas is 3 to 4.

4. With fig. and notation of Art. 217,

$$\text{area of pentagon}=5AD.OD=5r^2\tan 36°;$$

$$\therefore\ 250=5r^2\tan 36°;\ \pi r^2=\frac{22}{7}\times 50\cot 36°=216\cdot23\text{ sq. ft.}$$

5. Area of circle $=\pi r^2=1386$; whence $r=21$.

By Art. 214, perimeter $=16r\sin 22°\,30'=16\times 21\times\cdot382=128\cdot352$ in.

6. Area of circle $=\pi r^2=616$; whence $r=14$.

By Art. 215, perimeter of pentagon $=2.5.14\tan 36°=140\times\cdot727$

$=101\cdot78$ ft.

7. Area of circle $=\pi r^2=2464$; whence $r=28$.

Now in Art. 214, if AB is a side of the quindecagon, $OD=28$, and diameter of required circle $=2AO=2OD\sec 12°=2\times 28\times 1\cdot022=57\cdot232$ ft.

8. Let r be the radius of the circle; then

for the pentagon, $$50 = \frac{5}{2} r^2 \sin \frac{2\pi}{5},$$

for the dodecagon, $$\text{area} = \frac{12}{2} r^2 \sin \frac{\pi}{6}; \qquad \text{[Art. 214]}$$

$$\therefore \frac{\text{area of dodecagon}}{50} = \frac{12}{5} \cdot \frac{\sin 30^\circ}{\sin 72^\circ};$$

$$\therefore \text{ area of dodecagon} = 60 \operatorname{cosec} 72^\circ = 60 \times 1{\cdot}0515 = 63{\cdot}09 \text{ sq. ft.}$$

9. Let the perimeters of pentagon and decagon be denoted by $10a$ and $10b$ respectively. Then as in Example 2, page 209,

$$\text{area of pentagon} = 5a^2 \cot \frac{\pi}{5},$$

$$\text{area of decagon} = \frac{10b^2}{4} \cot \frac{\pi}{10};$$

$$\therefore 2a^2 \cot 36^\circ = b^2 \cot 18^\circ.$$

Now $$\cot^2 18^\circ = \frac{1+\cos 36^\circ}{1-\cos 36^\circ} = \frac{1+\frac{\sqrt{5}+1}{4}}{1-\frac{\sqrt{5}+1}{4}} = \frac{5+\sqrt{5}}{3-\sqrt{5}};$$

and $$\cot^2 36^\circ = \frac{1+\frac{\sqrt{5}-1}{4}}{1-\frac{\sqrt{5}-1}{4}} = \frac{3+\sqrt{5}}{5-\sqrt{5}};$$

$$\therefore \frac{\cot^2 18^\circ}{\cot^2 36^\circ} = \frac{20}{4} = 5.$$

$$\therefore \frac{a^2}{b^2} = \frac{\sqrt{5}}{2}, \text{ or } \frac{a}{b} = \frac{\sqrt[4]{5}}{\sqrt{2}}.$$

10. Let $2na$ be the common perimeter, so that $2a$, a are sides respectively of the two polygons.

Area of polygon of n sides $= \frac{n}{4} \cdot 4a^2 \cot \frac{\pi}{n}$.

Area of polygon of $2n$ sides $= \frac{2n}{4} \cdot a^2 \cot \frac{\pi}{2n}$;

$$\therefore \text{ ratio of areas} = \left(2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n}\right) : \left(\sin \frac{\pi}{n} \cos \frac{\pi}{2n}\right)$$

$$= \left(2 \cos \frac{\pi}{n} \sin \frac{\pi}{2n}\right) : \left(2 \sin \frac{\pi}{2n} \cos^2 \frac{\pi}{2n}\right)$$

$$= \left(2 \cos \frac{\pi}{n}\right) : \left(1 + \cos \frac{\pi}{n}\right).$$

11. In the fig. of Art. 214, if $AD=a$, $OA=R$, we have $R=a\operatorname{cosec}\frac{\pi}{n}$.

In the fig. of Art. 215, if $OD=r$, we have $r=a\cot\frac{\pi}{n}$.

$$\therefore\ R+r=a\left(\frac{1}{\sin\frac{\pi}{n}}+\frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right)$$

$$=\frac{a\left(1+\cos\frac{\pi}{n}\right)}{2\sin\frac{\pi}{2n}\cdot\cos\frac{\pi}{2n}}=\frac{2a\cos^2\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cdot\cos\frac{\pi}{2n}}=a\cot\frac{\pi}{2n}.$$

12. Let p, h, d represent a side of the pentagon, hexagon, and decagon respectively inscribed in a circle of radius r; then

$$p=2r\sin 36°,\quad h=2r\sin 30°,\quad d=2r\sin 18°.$$

$$\therefore\ h^2+d^2=4r^2\left\{\frac{1}{4}+\left(\frac{\sqrt{5}-1}{4}\right)^2\right\}=4r^2\left(\frac{4+6-2\sqrt{5}}{16}\right)$$

$$=4r^2\left(\frac{10-2\sqrt{5}}{16}\right)=4r^2\sin^2 36°=p^2.\qquad\text{[See Ex. Art. 126.]}$$

13. $A_1=\frac{1}{2}nr^2\sin\frac{2\pi}{n}$, $\qquad B_1=nr^2\tan\frac{\pi}{n}$, $\qquad$ [Arts. 214, 215]

$$A_2=\frac{1}{2}2n\,.\,r^2\sin\frac{\pi}{n},\qquad B_2=2nr^2\tan\frac{\pi}{2n}.$$

$$\therefore\ A_1B_1=\frac{1}{2}n^2r^4\,.\,2\sin\frac{\pi}{n}\cos\frac{\pi}{n}\tan\frac{\pi}{n}=n^2r^4\sin^2\frac{\pi}{n}=A_2^2.$$

Thus A_2 is the geom. mean between A_1 and B_1.

Again $\quad\frac{1}{A_2}+\frac{1}{B_1}=\frac{1}{nr^2\sin\frac{\pi}{n}}+\frac{1}{nr^2\tan\frac{\pi}{n}}=\frac{1+\cos\frac{\pi}{n}}{nr^2\sin\frac{\pi}{n}}$

$$=\frac{2\cos^2\frac{\pi}{2n}}{2nr^2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}=\frac{1}{nr^2\tan\frac{\pi}{2n}}=\frac{2}{B_2}.$$

Thus B_2 is the harm. mean between A_2 and B_1.

EXAMPLES. XVIII. c. PAGE 218.

1. Required distance

$$= r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}.$$

2. $I_1A = r_1 \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2}$,

$I_1B = r_1 \operatorname{cosec} \left(90° - \frac{B}{2}\right) = r_1 \sec \frac{B}{2} = 4R \sin \frac{A}{2} \cos \frac{C}{2}$,

$I_1C = r_1 \operatorname{cosec} \left(90° - \frac{C}{2}\right) = r_1 \sec \frac{C}{2} = 4R \sin \frac{A}{2} \cos \frac{B}{2}$.

3. (1) From the fig. of Art. 219, we have

$$\text{area} = \frac{1}{2} I_1I_2 \cdot I_3C = \frac{1}{2} 4R \cos \frac{C}{2} \cdot r_3 \operatorname{cosec} \frac{C}{2}$$

$$= 2Rr_3 \cot \frac{C}{2} = 2Rs.$$

(2) $\text{Area} = 2Rs = 2R \cdot \frac{\Delta}{r} = \frac{1}{2} \Delta \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}$. [Art. 212.]

4. We have $II_1 = IC \operatorname{cosec} II_1C = IC \operatorname{cosec} \frac{B}{2}$.

$$\therefore rII_1 \cdot II_2 \cdot II_3 = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cdot IA \cdot IB \cdot IC \cdot \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}$$

$$= 4R \cdot IA \cdot IB \cdot IC.$$

5. Perimeter of pedal triangle $= R(\sin 2A + \sin 2B + \sin 2C)$

$$= 4R \sin A \sin B \sin C.$$

In-radius of pedal triangle

$= 4 \cdot \frac{R}{2} \sin (90° - A) \sin (90° - B) \sin (90° - C)$ [Arts. 225, 212]

$= 2R \cos A \cos B \cos C.$

6. (1) $\frac{g}{a^2} + \frac{h}{b^2} + \frac{k}{c^2} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$$= \frac{2bc \cos A + 2ca \cos B + 2ab \cos C}{2abc}$$

$$= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc}.$$

(2) $\dfrac{b^2-c^2}{a^2}\,g+\ldots\ldots+\ldots\ldots=\dfrac{b^2-c^2}{a}\cos A+\ldots\ldots+\ldots\ldots$

$$=\frac{(b^2-c^2)\,(b^2+c^2-a^2)}{2abc}+\ldots\ldots+\ldots\ldots$$

$$=0.$$

7. Let ρ_1, ρ_2, ρ_3 be the radii, then

$$\rho_1=4\frac{R}{2}\cdot\sin\frac{G}{2}\cos\frac{H}{2}\cos\frac{K}{2}=2R\cos A\sin B\sin C. \qquad \text{[Arts. 225, 212.]}$$

8. (1) $a\cos A$, $b\cos B$, $c\cos C$, $180°-2A$, $180°-2B$, $180°-2C$ are the sides and angles of the pedal triangle of the triangle ABC. Hence in any formula connecting a, b, c, A, B, C, we may replace the sides by $a\cos A$, $b\cos B$, $c\cos C$ respectively, and the angles by $180°-2A$, $180°-2B$, $180°-2C$ respectively.

(2) $a\operatorname{cosec}\dfrac{A}{2}$, $b\operatorname{cosec}\dfrac{B}{2}$, $c\operatorname{cosec}\dfrac{C}{2}$, $90°-\dfrac{A}{2}$, $90°-\dfrac{B}{2}$, $90°-\dfrac{C}{2}$ are the sides and angles of the ex-central triangle of the triangle ABC. Hence the proposition follows.

9. We have $SI^2=R^2-2Rr$ [Art. 228].

$\therefore$ R^2-2Rr must be positive; that is $R-2r$ must be positive.

Hence R can never exceed $2r$.

10. If $R=2r$;

we have $SI^2=R^2-2Rr=0$;

$\therefore$ the centres of the circumcircle and in-circle coincide, and hence the triangle is equilateral.

11. $SI^2+SI_1^2+SI_2^2+SI_3^2=R^2-2Rr+R^2+2Rr_1+R^2+2Rr_2+R^2+2Rr_3$

$$=4R^2+2R\,(r_1+r_2+r_3-r)$$

$$=12R^2 \qquad \text{[XVIII. a. Ex. 25].}$$

12. (1) $a\,.\,AI^2+b\,.\,BI^2+c\,.\,CI^2=r^2a\operatorname{cosec}^2\dfrac{A}{2}+\ldots\ldots+\ldots\ldots$

$$=abcr^2\left\{\frac{1}{(s-b)\,(s-c)}+\ldots\ldots+\ldots\ldots\right\}$$

$$=\frac{abcr^2s}{(s-a)\,(s-b)\,(s-c)}=\frac{abcr^2s^2}{\Delta^2}=abc.$$

(2) $a\,.\,AI_1^2-b\,.\,BI_1^2-c\,.\,CI_1^2=r_1^2\left(a\operatorname{cosec}^2\dfrac{A}{2}-b\sec^2\dfrac{B}{2}-c\sec^2\dfrac{C}{2}\right)$

$$=abcr_1^2\left\{\frac{1}{(s-b)\,(s-c)}-\frac{1}{s\,(s-b)}-\frac{1}{s\,(s-c)}\right\}$$

$$=\frac{abcr_1^2\{s-(s-c)-(s-b)\}}{s\,(s-b)\,(s-c)}=\frac{abcr_1^2(s-a)^2}{\Delta^2}=abc.$$

13. (1) We have $\frac{OG}{AG} = \frac{\triangle OBC}{\triangle ABC}$.

$$\therefore \frac{OG}{AG} + \frac{OH}{BH} + \frac{OK}{CK} = \frac{\text{sum of areas of } \triangle^s OBC,\ OAC,\ OAB}{\triangle ABC} = 1.$$

(2) Since A, H, O, K are concyclic, $HK = AO \sin A$;

$$\text{thus } AO = \frac{a \cos A}{\sin A} = a \cot A.$$

$\therefore OG + a \cot A = OG + AO = AG$, and the result required is reduced to that already proved in (1).

14. Circum-radius of $\triangle AHK = \frac{KH}{2 \sin A} = \frac{R \sin 2A}{2 \sin A} = R \cos A$;

$\therefore$ sum of circum-radii of $\triangle^s AHK$, BKG, $CGH = R(\cos A + \cos B + \cos C)$

$$= R\left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right)$$

$$= R + r.$$

15. We have $A_1 = \frac{\pi}{2} - \frac{A}{2}$,

$$A_2 = \frac{\pi}{2} - \frac{A_1}{2} = \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \frac{A}{2}\right),$$

$$A_3 = \frac{\pi}{2} - \frac{A_2}{2} = \frac{\pi}{2} - \frac{1}{2}\left\{\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \frac{A}{2}\right)\right\};$$

.....

$$\therefore A_n = \frac{\pi}{2}\left(1 - \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \ldots \text{ to } n \text{ terms}\right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{2} \frac{1 - \left(-\frac{1}{2}\right)^n}{1 + \frac{1}{2}} + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3}\left(1 - (-1)^n \frac{1}{2^n}\right) + (-1)^n \frac{A}{2^n}$$

$$= \frac{\pi}{3} + (-1)^n \frac{1}{2^n}\left(A - \frac{\pi}{3}\right);$$

similarly $$B_n=\frac{\pi}{3}+(-1)^n\frac{1}{2^n}\left(B-\frac{\pi}{3}\right),$$

$$C_n=\frac{\pi}{3}+(-1)^n\frac{1}{2^n}\left(C-\frac{\pi}{3}\right).$$

When n is indefinitely increased, then $A_n=B_n=C_n=\frac{\pi}{3}$.

16. (1) $OS^2=R^2-8R^2\cos A\cos B\cos C$ [Art. 230]

$$=R^2+2R^2(1+\cos 2A+\cos 2B+\cos 2C)$$
$$=9R^2-2R^2(1-\cos 2A+1-\cos 2B+1-\cos 2C)$$
$$=9R^2-4R^2(\sin^2 A+\sin^2 B+\sin^2 C)$$
$$=9R^2-a^2-b^2-c^2.$$

(2) We have $AO=2R\cos A$; $IA=r\operatorname{cosec}\frac{A}{2}=4R\sin\frac{B}{2}\sin\frac{C}{2}$;

$$\angle IAO=(90^\circ-B)-\frac{A}{2}=\frac{C-B}{2};$$

$$\therefore\ OI^2=4R^2\cos^2 A+16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2}-16R^2\cos A\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$$

$$=4R^2\left(\cos^2 A+4\sin^2\frac{B}{2}\sin^2\frac{C}{2}-\cos A\sin B\sin C-4\cos A\sin^2\frac{B}{2}\sin^2\frac{C}{2}\right)$$

$$=4R^2\left(\cos^2 A+8\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}-\cos A\sin B\sin C\right)$$

$$=2r^2-4R^2\cos A(\sin B\sin C-\cos A)$$

$$=2r^2-4R^2\cos A\cos B\cos C.$$

(3) We have

$$AO=2R\cos A;\ I_1A=r_1\operatorname{cosec}\frac{A}{2}=4R\cos\frac{B}{2}\cos\frac{C}{2};\ \angle I_1AO=\frac{C-B}{2};$$

$$\therefore\ OI_1^2=4R^2\cos^2 A+16R^2\cos^2\frac{B}{2}\cos^2\frac{C}{2}-16R^2\cos A\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{C-B}{2}$$

$$=4R^2\left(\cos^2 A+4\cos^2\frac{B}{2}\cos^2\frac{C}{2}-4\cos A\cos^2\frac{B}{2}\cos^2\frac{C}{2}-\cos A\sin B\sin C\right)$$

$$=4R^2\left(\cos^2 A+8\sin^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}-\cos A\sin B\sin C\right)$$

$$=2r_1^2-4R^2\cos A(\sin B\sin C-\cos A)$$

$$=2r_1^2-4R^2\cos A\cos B\cos C.$$

17. Let R', ρ, ρ_1, ρ_2, ρ_3 represent the circum-radius, the in-radius, and the ex-radii of the pedal triangle; then, since A, B, C are the ex-centres of the pedal triangle, we have, as in Art. 229,

$$f^2=R'^2+2R'\rho_1;\ \text{also } R'=\frac{R}{2}.$$

$$\therefore f^2+g^2+h^2=3R'^2+2R'(\rho_1+\rho_2+\rho_3)$$

$$=3R'^2+2R'(4R'+\rho) \qquad \text{[XVIII. a. Ex. 25]}$$

$$=11R'^2+2R'\rho$$

$$=11R'^2+8R'^2\sin\frac{G}{2}\sin\frac{H}{2}\sin\frac{F}{2}$$

$$=11R'^2+8R'^2\cos A\cos B\cos C; \qquad \text{[Art. 224]}$$

$$\therefore\ 4(f^2+g^2+h^2)=11R^2+8R^2\cos A\cos B\cos C. \qquad \text{[Art. 225.]}$$

EXAMPLES. XVIII. d. PAGE 223.

1. Let r be the radius, and a, b, c, d the sides of the quadrilateral.

Then we have $2S=ra+rb+rc+rd$;

$$\therefore\ r=\frac{S}{\sigma}.$$

2. Let $ABCD$ be the quadrilateral having sides

$$AB=BC=3;\ \ CD=DA=4.$$

Then $$\angle BAC=\angle BCA;\ \ \angle DAC=\angle DCA;$$

$$\angle BAD+\angle BCD=180^\circ;\ \ \therefore\ \angle BAD=\angle BCD=90^\circ,$$

and the $\Delta^s BAD$, BCD are identically equal.

Thus it easily follows that the bisectors of the $\angle^s BAD$, BCD meet on BD which bisects the angles ABC, ADC.

$\therefore$ a circle can be inscribed in the quadrilateral.

If r be its radius we have

$$r(3+3+4+4)=2\ (\text{area of quadrilateral})=2\,.\,4\,.\,3.$$

$$\therefore\ r=\frac{12}{7}=1\tfrac{5}{7}.$$

Also radius of circumscribed circle $=\frac{1}{2}BD=\frac{1}{2}\sqrt{3^2+4^2}=\frac{5}{2}=2\frac{1}{2}$.

3. Let $ABCD$ be the quadrilateral, and let

$$AB=1,\ BC=2,\ CD=4,\ DA=3.$$

Then $$AC^2=5-4\cos ABC=5+4\cos ADC;$$

also $$AC^2=25-24\cos ADC;$$

$$\therefore\ 5+4\cos ADC=25-24\cos ADC.$$

$$\therefore\ \cos ADC=\frac{20}{28}=\frac{5}{7};$$

and area of quadrilateral $=\sqrt{4\times3\times1\times2}=\sqrt{24}$;

$$\therefore\ \text{radius of inscribed circle}=\frac{\sqrt{24}}{5}=\cdot98 \text{ nearly.}$$

4. Let $ABCD$ be the quadrilateral, and let

$$AB=60,\ BC=25,\ CD=52,\ DA=39.$$

Then $AC^2=60^2+25^2-2\,.\,60\,.\,25\cos ABC=5^2\times13^2-2\times60\times25\cos ABC$;

also $$AC^2=52^2+39^2-2\times52\times39\cos ADC=5^2\times13^2+2\times52\times39\cos ABC;$$

$$\therefore\ \cos ABC=0, \text{ that is } \angle ABC=90°;$$

and hence the $\angle ADC=90°$;

$$\therefore\ AC^2=60^2+25^2=5^2\times13^2;\ \therefore\ AC=65;$$

$$\therefore\ BD=\frac{60\times52+39\times25}{65}=48+15=63.$$

Also the area $=\sqrt{28\times63\times36\times49}=6^2\times7^2=1764.$

5. Let $ABCD$ be the quadrilateral, and let

$$AB=4,\ BC=5,\ CD=8,\ DA=9.$$

Then since $AB+BC=9$, AC must be less than 9;

$$\therefore \text{ diagonal } BD=9;$$

and $$\cos A=\frac{2}{9};\ \ \therefore\ \sin A=\frac{\sqrt{77}}{9};$$

$$\cos C=\frac{1}{10};\ \ \therefore\ \sin C=\frac{3\sqrt{11}}{10};$$

$$\therefore\ \text{area}=\frac{1}{2}(ad\sin A+bc\sin C)=18\frac{\sqrt{77}}{9}+20\,.\,\frac{3\sqrt{11}}{10}=2\sqrt{77}+6\sqrt{11}.$$

6. Since the quadrilateral is such that one circle can be inscribed in it and another circle circumscribed about it, therefore

$$\cos A = \frac{ad-bc}{ad+bc}, \text{ and } \sin A = \frac{2\sqrt{abcd}}{ad+bc}; \qquad \text{[Art. 234, Ex.]}$$

$$\therefore \text{ area} = \frac{1}{2}(ad \sin A + bc \sin C) = \sqrt{abcd}.$$

If r be radius of inscribed circle,

$$\frac{r}{2}(a+b+c+d) = \text{area} = \sqrt{abcd}.$$

$$\therefore \ r = \frac{2\sqrt{abcd}}{a+b+c+d}.$$

7. We have $S^2 = (\sigma-a)(\sigma-b)(\sigma-c)(\sigma-d) - abcd\cos^2\alpha$; [Art. 232]

$\therefore$ S is greatest when $\cos\alpha = 0$, since σ, a, b, c, d are constant.

But α is half the sum of two opposite angles;

$\therefore$ area is a maximum when the sum of two opposite angles is 180°;

that is, when the quadrilateral can be inscribed in a circle.

8. $23+29+37+41 = 130$;

$\therefore$ maximum area $= \sqrt{(65-23)(65-29)(65-37)(65-41)}$ sq. inches

$= 6\times 6\times 4\times 7$ sq. inches $= 7$ sq. feet.

9. We have $\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$; [Art. 233]

$$\therefore \tan^2\frac{B}{2} = \frac{1-\cos B}{1+\cos B} = \frac{(c+d)^2-(a-b)^2}{(a+b)^2-(c-d)^2}$$

$$= \frac{(a+c+d-b)(c+d+b-a)}{(a+b+c-d)(a+b+d-c)} = \frac{(\sigma-a)(\sigma-b)}{(\sigma-c)(\sigma-d)}.$$

10. See figure on page 220. Let $\angle DPA = \beta$;

then

$$a^2 = AP^2 + PB^2 + 2AP \cdot PB\cos\beta;$$

$$c^2 = DP^2 + PC^2 + 2DP \cdot PC\cos\beta;$$

$$b^2 = BP^2 + PC^2 - 2BP \cdot PC\cos\beta;$$

$$d^2 = DP^2 + PA^2 - 2DP \cdot PA\cos\beta;$$

$$\therefore \ (a^2+c^2) - (b^2+d^2) = 2\cos\beta\{AP(DP+PB) + PC(DP+PB)\}$$

$$= 2AC \cdot BD\cos\beta = 2fg\cos\beta.$$

11. Area $= \frac{1}{2} AC \cdot BD \sin\beta$ [Art. 231]

$= \frac{1}{4}\{(a^2+c^2) \sim (b^2+d^2)\} \tan\beta.$ [Ex. 10.]

12. We have $a+c=b+d$; $\therefore\ (a-d)^2=(b-c)^2$.

Also $a^2+d^2-2ad\cos A = b^2+c^2-2bc\cos C$;

$\therefore$ by subtraction, $ad(1-\cos A)=bc(1-\cos C)$;

that is, $ad-bc=ad\cos A - bc\cos C$;

$\therefore\ a^2d^2(1-\cos^2 A)+b^2c^2(1-\cos^2 C)=2abcd-2abcd\cos A\cos C$;

or $a^2d^2\sin^2 A + b^2c^2\sin^2 C = 2abcd - 2abcd\cos A\cos C$.

$\therefore\ (2S)^2 - 2abcd\sin A\sin C = 2abcd - 2abcd\cos A\cos C$.

$\therefore\ 4S^2 = 2abcd(1-\cos\overline{A+C}) = 4abcd\sin^2\frac{A+C}{2}$.

$\therefore\ S = \sqrt{abcd}\sin\frac{A+C}{2}$.

13. If β be the angle between the diagonals, we have

$$S = \frac{1}{2} fg\sin\beta.$$

$\therefore\ f^2g^2 - 4S^2 = f^2g^2\cos^2\beta$

$= \frac{1}{4}\{(a^2+c^2)-(b^2+d^2)\}^2$ [Ex. 10]

$= \frac{1}{4}(2bd-2ac)^2$, since $a+c=b+d$;

that is, $4S^2 = f^2g^2 - (ac-bd)^2$.

14. (1) By Euc. VI. D, we have

$$(ac+bd)\sin\beta = fg\sin\beta = 2S = (ad+bc)\sin A.$$

(2) $\cos\beta = \frac{(a^2+c^2)\sim(b^2+d^2)}{2fg}$ [Ex. 10]

$= \frac{(a^2+c^2)\sim(b^2+d^2)}{2(ac+bd)}$.

(3) $\tan^2\frac{\beta}{2}=\frac{1-\cos\beta}{1+\cos\beta}=\frac{(b+d)^2-(c-a)^2}{(c+a)^2-(b-d)^2}$, or $\frac{(c+a)^2-(b-d)^2}{(b+d)^2-(c-a)^2}$

$=\frac{(b+d+c-a)(b+d-c+a)}{(c+a-b+d)(c+a+b-d)}$, or $\frac{(c+a-b+d)(c+a+b-d)}{(b+d+c-a)(b+d-c+a)}$

$=\frac{(\sigma-a)(\sigma-c)}{(\sigma-b)(\sigma-d)}$, or $\frac{(\sigma-b)(\sigma-d)}{(\sigma-a)(\sigma-c)}$.

15. $S=\frac{1}{2}fg\sin\beta$

$=\frac{1}{4}\sqrt{4f^2g^2-4f^2g^2\cos^2\beta}$

$=\frac{1}{4}\sqrt{4f^2g^2-(a^2+c^2-b^2-d^2)^2}$. [Ex. 10.]

16. See figure on page 220.

We have $\frac{AP}{PC}=\frac{\triangle APB}{\triangle CPB}=\frac{\triangle APD}{\triangle CPD}=\frac{\triangle DAB}{\triangle DCB}=\frac{ad}{bc}$;

$\therefore\ AP=\frac{ad}{ad+bc}\cdot AC$;

$PC=\frac{bc}{ad+bc}\cdot AC$;

$\therefore\ AP\cdot PC=\frac{abcd}{(ad+bc)^2}\cdot AC^2=\frac{abcd}{(ad+bc)^2}\cdot\frac{(ad+bc)(ac+bd)}{ab+cd}$

$=\frac{abcd(ac+bd)}{(ab+cd)(ad+bc)}$.

EXAMPLES. XVIII. e. PAGE 225.

1. $242+1212+1450=2904$;

$\therefore$ area $=\sqrt{1452\times1210\times240\times2}$ sq. yds.

$=8\times3\times121\times10$ sq. yds.

$=\frac{8\times3\times121\times10}{4840}=6$ acres.

2. Area $=\frac{200^2\sin22\frac{1}{2}^\circ\sin67\frac{1}{2}^\circ}{2\sin(22\frac{1}{2}^\circ+67\frac{1}{2}^\circ)}=\frac{200^2\cos45^\circ}{4}$

$=\frac{10000\sqrt{2}}{2}=\frac{14142}{2}=7071$ sq. yds.

3. We have $\frac{\Delta}{s-a}=\frac{2\Delta}{s-b}=\frac{2\Delta}{s-c}$;

$$\therefore\ b=c, \text{ and } s-b=2(s-a);$$

$$\therefore\ \frac{c+a-b}{2}=b+c-a;$$

that is, $$\frac{a}{2}=2b-a;$$

that is, $$3a=4b.$$

4. If a, b, c are in A.P., we have $a+c=2b$;

$$\therefore\ \frac{1}{r_1}+\frac{1}{r_3}=\frac{s-a+s-c}{\Delta}=\frac{2(s-b)}{\Delta}=\frac{2}{r_3};$$

$$\therefore\ r_1, r_2, r_3 \text{ are in H.P.}$$

5. We have $s=\frac{z}{x}+\frac{x}{y}+\frac{y}{z}$.

$$\therefore\ \text{area}=\sqrt{\left(\frac{z}{x}+\frac{x}{y}+\frac{y}{z}\right)\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}}=\sqrt{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}.$$

6. We have $\frac{\sin(A-B)}{\sin(B-C)}=\frac{\sin A}{\sin C}=\frac{\sin(B+C)}{\sin(A+B)}$;

$$\therefore\ \sin(A-B)\sin(A+B)=\sin(B-C)\sin(B+C);$$

or $$\sin^2A-\sin^2B=\sin^2B-\sin^2C;$$

$$\therefore\ a^2-b^2=b^2-c^2;$$

that is, a^2, b^2, c^2 are in A.P.

7. First side $=\frac{a\sin A+b\sin B+c\sin C}{\sin A+\sin B+\sin C}=\frac{a^2+b^2+c^2}{a+b+c}=\frac{a^2+b^2+c^2}{2s}$.

8. First side $=\frac{a}{\sin A}\cdot(a+b+c)\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$=\frac{abc}{2\Delta}\cdot 2s\frac{(s-a)(s-b)(s-c)}{abc}=\frac{\Delta^2}{\Delta}=\Delta.$$

9. By Ex. 20, XVIII. a. we have

first side $=abc\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$$=abc\sqrt{\frac{s(s-a)}{(s-b)(s-c)}\cdot\frac{s(s-b)}{(s-c)(s-a)}\cdot\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\frac{abcs^2}{\Delta}=4Rs^2=4R(r_2r_3+r_3r_1+r_1r_2). \qquad \text{[XVIII. a. Ex. 24.]}$$

10. $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{r_1}{s}+\frac{r_2}{s}+\frac{r_3}{s}=\frac{r_1+r_2+r_3}{s}$

$$=\frac{r_1+r_2+r_3}{(r_2r_3+r_3r_1+r_1r_2)^{\frac{1}{2}}}.$$ [XVIII. a. Ex. 24.]

11. First side $= bc\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+ca\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}+ab\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$$=\frac{s}{\Delta}\{bc(s-a)+ca(s-b)+ab(s-c)\}$$

$$=\frac{s}{\Delta}\{(bc+ca+ab)s-3abc\}$$

$$=\frac{abcs^2}{\Delta}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{s}\right)$$

$$=4Rs^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{s}\right).$$

12. First side $=\left(\frac{s+s-a+s-b+s-c}{\Delta}\right)^2=\left(\frac{2s}{\Delta}\right)^2$

$$=\frac{4s}{\Delta}\cdot\left(\frac{s-a+s-b+s-c}{\Delta}\right)=\frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right).$$

13. We have $\frac{\Delta}{s}=6$, $2s=70$; $\therefore\ \Delta=6\times35$.

Let a, b be the sides containing the right angle;

then $$\frac{1}{2}ab=\Delta=6\times35;\quad\therefore\ ab=12\times35;$$

also $$a^2+b^2=c^2=\{70-(a+b)\}^2;$$

$$\therefore\ 0=70^2-140(a+b)+2ab=70^2-140(a+b)+12\times70;$$

$$\therefore\ a+b=41=20+21,$$

and $$ab=12\times35=20\times21;$$

$\therefore$ the two sides containing the right angle are 20, 21, and the hypotenuse

$$=70-41=29.$$

14. It is easily seen from a figure that $\frac{a}{2f}=\tan A$;

$$\therefore\ \frac{a}{f}+\frac{b}{g}+\frac{c}{h}=2(\tan A+\tan B+\tan C)$$

$$=2\tan A\tan B\tan C=\frac{abc}{4fgh}.$$

15. Let r, r' be the radii of the inscribed circles; then since the perimeters of the triangle and hexagon are equal we have

$$6r\tan\frac{\pi}{3}=12r'\tan\frac{\pi}{6};$$

whence

$$3r=2r';$$

$$\therefore\ \frac{\pi r^2}{\pi r'^2}=\frac{4}{9};$$

$\therefore$ areas of inscribed circles are as 4 to 9.

16. Perimeter $=R(\sin 2A+\sin 2B+\sin 2C)=4R\sin A\sin B\sin C$

$$=4R\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}=\frac{abc}{2R^2}.$$

17. Area

$$=\frac{1}{2}4R\cos\frac{B}{2}4R\cos\frac{C}{2}\sin\left(90^\circ-\frac{A}{2}\right)$$

$$=8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$=2R^2(\sin A+\sin B+\sin C)$$

$$=R(a+b+c)=\frac{abc(a+b+c)}{4\Delta}.$$

18. Let O_1, O_2 be the two circumcentres; then O_1O_2 is at right angles to AC at its middle point. Draw O_1N_1, O_2N_2 perpendicular to AB_1;

then

$$O_1O_2=N_1N_2\operatorname{cosec}A=\frac{c_1\sim c_2}{2\sin A}.$$

19. Let f, g be the diagonals; then by Euc. VI. D.,

$$(ac+bd)\sin\beta=fg\sin\beta=2S. \qquad \text{[Art. 231.]}$$

$$\therefore\ \sin\beta=\frac{2S}{ac+bd}.$$

20. We have $r\,.\,II_1=r\,.\,IC\operatorname{cosec}II_1C=ICr\operatorname{cosec}\frac{B}{2}=IC\,.\,IB.$

$$\therefore\ r^3II_1\,.\,II_2\,.\,II_3=IA^2\,.\,IB^2\,.\,IC^2.$$

21. Sum of squares of sides of ex-central Δ

$$=16R^2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)$$

$$=16R^2\left(2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) \qquad \text{[XII. d. Ex. 13]}$$

$$=32R^2+32R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=32R^2+8Rr=8R(4R+r).$$

22. Since a circle can be inscribed in the quadrilateral,

$$\therefore\ a+c=b+d;$$

and since a circle can be circumscribed about the quadrilateral,

$$\therefore\ \cos\beta=\frac{(a^2+c^2)\sim(b^2+d^2)}{2(ac+bd)} \qquad \text{[XVIII. } d\text{. Ex. 14]}$$

$$=\frac{(a+c)^2\sim(b+d)^2+2(ac\sim bd)}{2(ac+bd)}$$

$$=\frac{ac\sim bd}{ac+bd}.$$

23. (1) We have $2l^2+2.\frac{a^2}{4}=b^2+c^2;$

$$2m^2+2.\frac{b^2}{4}=c^2+a^2;$$

$$2n^2+2.\frac{c^2}{4}=a^2+b^2;$$

$\therefore$ by addition, $4(l^2+m^2+n^2)=3(a^2+b^2+c^2).$

(2) First side $=\frac{1}{4}\{(b^2-c^2)(2b^2+2c^2-a^2)+\text{two similar terms}\}$

$$=\frac{1}{4}[\{2(b^4-c^4)-a^2(b^2-c^2)\}+\ldots+\ldots]$$

$$=0.$$

(3) $4l^2=2b^2+2c^2-a^2;$

$$\therefore\ 16(l^4+m^4+n^4)=(2b^2+2c^2-a^2)^2+\ldots+\ldots$$

$$=9(a^4+b^4+c^4),\ \text{on reduction.}$$

24. We have $r_1+r_2+r_3=4R+r$ [XVIII. a. Ex. 25],

$r_1r_2+r_2r_3+r_3r_1=s^2$ [XVIII. a. Ex. 24],

$r_1r_2r_3=rs^2;$ [XVIII. a. Ex. 15]

$\therefore$ the equation whose roots are r_1, r_2, r_3 is

$$x^3-(4R+r)x^2+s^2x-rs^2=0.$$

25. Let PQ be the tangent to inscribed circle parallel to BC, and draw AHD at right angles to PQ and BC;

then $\frac{\Delta_1}{\Delta}=\frac{AP.AQ}{AB.AC}=\frac{AH}{AD}\cdot\frac{AH}{AD}=\left(\frac{AD-2r}{AD}\right)^2$

$$=\left(1-\frac{2\Delta}{s}\cdot\frac{a}{2\Delta}\right)^2=\frac{(s-a)^2}{s^2}.$$

$$\therefore\ \frac{\Delta_1}{(s-a)^2}=\frac{\Delta}{s^2}$$

$$=\frac{\Delta_2}{(s-b)^2}=\frac{\Delta_3}{(s-c)^2},\ \text{by symmetry.}$$

Otherwise. Let the sides and perimeter of triangle APQ be denoted by $a_1, b_1, c_1, 2s_1$; then, by Art. 213, $s_1 = s - a$.

But $$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = \frac{s}{s_1}; \quad \therefore \frac{\Delta_1}{s_1^2} = \frac{\Delta}{s^2}.$$ [Euc. VI. 19].

That is, $$\frac{\Delta_1}{(s-a)^2} = \frac{\Delta}{s^2}.$$

26. MN is perpendicular to the bisector of the angle A;

$$\therefore MN \text{ is parallel to } I_2I_3;$$

thus the sides of the $\triangle LMN$ are parallel to the sides of the excentral $\triangle$ and the triangles are similar.

Also $$MN = 2(s-a)\sin\frac{A}{2};$$

$$\therefore \frac{\triangle LMN}{\triangle I_1I_2I_3} = \frac{MN \cdot NL}{I_2I_3 \cdot I_3I_1} = \frac{2(s-a)\sin\frac{A}{2} \cdot 2(s-b)\sin\frac{B}{2}}{4R\cos\frac{A}{2} \cdot 4R\cos\frac{B}{2}}$$

$$= \frac{(s-a)(s-b)(s-c)}{4R^2 s} = \frac{\Delta^2}{4R^2 s^2} = \frac{r^2}{4R^2}.$$

27. We have $\angle PBC = \angle PCB = A$; $\therefore \angle QPR = 180° - 2A$.

Similarly $$\angle PQR = 180° - 2B, \ \angle QRP = 180° - 2C.$$

Again $$AQ = \frac{AC\sin B}{\sin AQC} = \frac{b\sin B}{\sin 2B} = \frac{b}{2\cos B};$$

similarly $$AR = \frac{c}{2\cos C}.$$

$$\therefore QR = \frac{b}{2\cos B} + \frac{c}{2\cos C} = \frac{b\cos C + c\cos B}{2\cos B\cos C} = \frac{a}{2\cos B\cos C}.$$

28. (1) We have $pc\sin\frac{A}{2} + pb\sin\frac{A}{2} = 2\Delta = bc\sin A$;

$$\therefore p(b+c) = 2bc\cos\frac{A}{2};$$

$$\therefore \frac{1}{p}\cos\frac{A}{2} = \frac{b+c}{2bc} = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right);$$

similarly $$\frac{1}{q}\cos\frac{B}{2} = \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right), \quad \frac{1}{r}\cos\frac{C}{2} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

$$\therefore \frac{1}{p}\cos\frac{A}{2} + \frac{1}{q}\cos\frac{B}{2} + \frac{1}{r}\cos\frac{C}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(2) We have $$pqr = \frac{8a^2b^2c^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{(b+c)(c+a)(a+b)}$$

$$= \frac{2a^2b^2c^2(\sin A + \sin B + \sin C)}{(b+c)(c+a)(a+b)}$$

$$= \frac{4\Delta\, abc\,(a+b+c)}{(b+c)(c+a)(a+b)};$$

$$\therefore \frac{pqr}{4\Delta} = \frac{abc\,(a+b+c)}{(b+c)(c+a)(a+b)}.$$

29. Let O be the orthocentre; then G, H, K are the middle points of OL, OM, ON respectively; therefore the sides of the ΔLMN are double the sides of the pedal triangle and parallel to them.

Therefore also the angles are equal to the angles of the pedal triangle.

(1) Area of $\Delta LMN = \frac{1}{2} \cdot 2a\cos A \cdot 2b\cos B \cdot \sin(180° - 2C)$

$$= 4ab\sin C \cdot \cos A\cos B\cos C = 8\Delta\cos A\cos B\cos C.$$

(2) $AL = AG + OG = c\sin B + c\cos B\cot C$

$$= \frac{c\cos(B-C)}{\sin C} = \frac{a\cos(B-C)}{\sin A};$$

$$\therefore AL\sin A + BM\sin B + CN\sin C = a\cos(B-C) + b\cos(C-A) + c\cos(A-B)$$

$$= 2R\{\sin\overline{B+C}\cos\overline{B-C} + \ldots + \ldots\}$$

$$= R[(\sin 2B + \sin 2C) + \ldots + \ldots]$$

$$= 2R[\sin 2A + \sin 2B + \sin 2C]$$

$$= 8R\sin A\sin B\sin C.$$

30. Let P be the centre of the circle inscribed between the in-circle and the sides AB, AC; then

(1) $$\frac{r - r_a}{r + r_a} = \sin\frac{A}{2};$$

$$\therefore r_a = r \cdot \frac{1 - \sin\frac{A}{2}}{1 + \sin\frac{A}{2}} = r\tan^2\frac{\pi - A}{4}.$$ [Compare XI. f. Ex. 15.]

(1) Also $$\frac{\pi - A}{4} + \frac{\pi - B}{4} + \frac{\pi - C}{4} = \frac{\pi}{2};$$

$$\therefore \sqrt{r_br_c} + \sqrt{r_cr_a} + \sqrt{r_ar_b} = r\left(\tan\frac{\pi - B}{4}\tan\frac{\pi - C}{4} + \ldots + \ldots\right)$$

$$= r.$$

31. It is easily seen that the triangle XYZ is the same as the triangle PQR in Ex. 27.

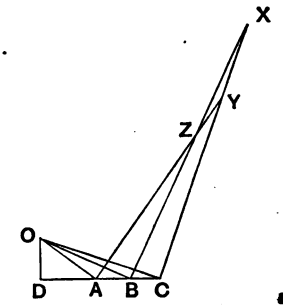
$$\therefore \text{Perimeter} = \frac{a}{2\cos B\cos C} + \frac{b}{2\cos C\cos A} + \frac{c}{2\cos A\cos B}$$

$$= \frac{R(\sin 2A + \sin 2B + \sin 2C)}{2\cos A\cos B\cos C} = 2R\tan A\tan B\tan C.$$

$$\text{Area} = \frac{1}{2}\cdot\frac{a}{2\cos B\cos C}\cdot\frac{b}{2\cos C\cos A}\cdot\sin 2C$$

$$= \frac{ab\sin C}{4\cos A\cos B\cos C} = R^2\tan A\tan B\tan C.$$

32. Let XYZ be the triangle formed by the tangents, and let the perpendiculars from X, Y, Z to the chord be represented by x, y, z respectively.



Then the area of

$$\triangle XYZ = \triangle BXC - \triangle ACY + \triangle AZB$$

$$= \frac{1}{2}x\,.\,BC - \frac{1}{2}y\,.\,AC + \frac{1}{2}z\,.\,AB.$$

Now $\frac{x}{CX} = \frac{DC}{CO}$, since OCX is a right angle.

Also $\frac{CX}{CO} = \frac{DB}{OD} = \frac{DB}{p}$, since O, B, C, X are concyclic.

$$\therefore x = \frac{DC\,.\,DB}{p}.$$

$$\therefore \triangle BXC = \frac{1}{2p}\,.\,DC\,.\,DB\,(DC - DB) = \frac{cb\,(c-b)}{2p},$$ if a, b, c denote the distances of A, B, C from D.

$$\therefore \text{area of } \triangle XYZ = \frac{cb\,(c-b) - ca\,(c-a) + ba\,(b-a)}{2p}$$

$$= \frac{cb\,(c-b) + ba\,(b-a) + ac\,(a-c)}{2p}$$

$$= -\frac{(c-b)\,(b-a)\,(a-c)}{2p} = \frac{BC\,.\,CA\,.\,AB}{2p},$$

for $CA = c - a = -(a-c)$.

MISCELLANEOUS EXAMPLES. F. PAGE 228.

1. We have $\cos(\alpha+\beta)=\cos(180°-\overline{\gamma+\delta})=-\cos(\gamma+\delta)$;

$\therefore \cos\alpha\cos\beta-\sin\alpha\sin\beta=\sin\gamma\sin\delta-\cos\gamma\cos\delta$;

$\therefore \cos\alpha\cos\beta+\cos\gamma\cos\delta=\sin\alpha\sin\beta+\sin\gamma\sin\delta$.

2. First side $=\dfrac{\cos(15°-A)\sin 15°-\sin(15°-A)\cos 15°}{\sin 15°\cos 15°}$

$=\dfrac{2\sin A}{\sin 30°}=4\sin A$.

3. $\cot A+\sin A\operatorname{cosec} B\operatorname{cosec} C=\dfrac{\cos A\sin B\sin C+1-\cos^2 A}{\sin A\sin B\sin C}$

$=\dfrac{\cos A\{\sin B\sin C+\cos(B+C)\}+1}{\sin A\sin B\sin C}$

$=\dfrac{\cos A\cos B\cos C+1}{\sin A\sin B\sin C}$,

which is symmetrical with respect to A, B, C.

4. We have $\sin B=\dfrac{b\sin A}{a}=\dfrac{\sqrt{8}\,.\sin 30°}{2}=\dfrac{1}{\sqrt{2}}$;

$\therefore B=45°$, or $135°$; [Art. 148. (iii)];

$\therefore C=105°$, or $15°$;

$\therefore c=\dfrac{a\sin C}{\sin A}=4\cos 15°$, or $4\sin 15°$;

$\therefore c=\sqrt{6}+\sqrt{2}$, or $\sqrt{6}-\sqrt{2}$.

5. (1) We have $\cot 18°\tan 36°=\dfrac{2\cos 18°\sin 36°}{2\sin 18°\cos 36°}$

$=\dfrac{\sin 54°+\sin 18°}{\sin 54°-\sin 18°}$

$=\dfrac{\sqrt{5}+1+(\sqrt{5}-1)}{\sqrt{5}+1-(\sqrt{5}-1)}=\sqrt{5}$.

(2) $\sin 36°=\sin 144°$, and $\sin 72°=\sin 108°$;

$\therefore$ first side $=4(2\sin 72°\sin 36°)^2=4(\cos 36°-\cos 108°)^2$

$=4(\cos 36°+\sin 18°)^2=4\left(\dfrac{\sqrt{5}+1}{4}+\dfrac{\sqrt{5}-1}{4}\right)^2$

$=5$.

6. $\log 2 = \cdot 30103$; $\therefore \log 4 = \cdot 60206$

$\log 3 = \cdot 47712$; $\therefore \log 9 = \cdot 95424$

$\log 11 = 1 \cdot 04139$

$\overline{2} \cdot 59769$

$\therefore \log \cdot 0396 = \overline{2} \cdot 59769$;

$\therefore \log (\cdot 0396)^{90} = -180 + 53 \cdot 7921 = \overline{127} \cdot 7921$;

$\therefore$ number of ciphers before the first significant digit in $(\cdot 0396)^{90}$ is 126.

7. Let P, Q be the two positions of the observer;

then $\angle QPB = 30°$, $\angle QBP = 45°$, $PQ = 50$ yards;

$$\therefore PB = \frac{50 \sin 75°}{\sin 45°} = \frac{50\sqrt{2}(\sqrt{3}+1)}{2\sqrt{2}} = 25(\sqrt{3}+1) = 68 \cdot 3 \text{ yds.},$$

$$QB = \frac{50 \sin 30°}{\sin 45°} = 50\sqrt{2} \cdot \frac{1}{2} = 35 \cdot 35 \text{ yds.}$$

8. $\text{First side} = 2 + \frac{1}{2}\{\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2(\alpha+\beta+\gamma)\}$

$= 2 + \cos(\alpha+\beta)\cos(\alpha-\beta) + \cos(\alpha+\beta+2\gamma)\cos(\alpha+\beta)$

$= 2 + 2\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha)$.

9. (1) $\tan 40° + \cot 40° = \dfrac{\sin^2 40° + \cos^2 40°}{\cos 40° \sin 40°} = \dfrac{2}{2 \sin 40° \cos 40°}$

$$= \frac{2}{\sin 80°} = 2 \sec 10°.$$

(2) $\tan 70° + \tan 20° = \tan 20° + \cot 20° = \dfrac{2}{\sin 40°}$, as in (1),

$= 2 \operatorname{cosec} 40°$.

10. (1) $\text{First side} = 2 \sin 4\alpha - 2 \cos 6\alpha \sin 4\alpha$

$= 2 \sin 4\alpha (1 - \cos 6\alpha) = 4 \sin 4\alpha \sin^2 3\alpha$

$= 16 \sin \alpha \cos \alpha \cos 2\alpha \sin^2 3\alpha$.

(2) $\text{First side} = \sin \dfrac{2\pi}{7} - 2 \cos \dfrac{5\pi}{7} \sin \dfrac{\pi}{7}$

$$= 2 \sin \frac{\pi}{7}\left(\cos \frac{\pi}{7} - \cos \frac{5\pi}{7}\right)$$

$$= 4 \sin \frac{\pi}{7} \sin \frac{2\pi}{7} \sin \frac{3\pi}{7}$$

$$= 4 \sin \frac{\pi}{7} \sin \frac{5\pi}{7} \sin \frac{3\pi}{7}.$$

11. We have $\sin C = \dfrac{c \sin B}{b} = \dfrac{6-2\sqrt{3}}{2(3\sqrt{2}-\sqrt{6})} = \dfrac{1}{\sqrt{2}}$;

$\therefore\ C = 45°$ or $135°$; [Art. 148, (iii)];

$\therefore\ A = 105°$ or $15°$;

$$\therefore\ a = \frac{c \sin A}{\sin C} = \frac{(6-2\sqrt{3})\sqrt{2}(\sqrt{3}+1)}{2\sqrt{2}}, \text{ or } \frac{(6-2\sqrt{3})\sqrt{2}(\sqrt{3}-1)}{2\sqrt{2}};$$

that is, $a = 2\sqrt{3}$, or $4\sqrt{3}-6$.

12. Let C be the rock and A, B the two positions of the ship.

Then we have $\angle BAC = \angle BCA = 67\frac{1}{2}°$;

$\therefore\ BC = BA = 10$ miles,

and $AC = 2AB \sin 22\frac{1}{2}° = 10\sqrt{2-\sqrt{2}}$ miles. [Art. 251.]

13. First side $= \dfrac{\sin^2 B - \sin^2 C}{\cos B + \cos C} + \ldots + \ldots$

$= \dfrac{\cos^2 C - \cos^2 B}{\cos B + \cos C} + \ldots + \ldots$

$= (\cos C - \cos B) + \ldots + \ldots$

$= 0.$

14. We have $\dfrac{1}{\cos(\theta-\alpha)} + \dfrac{1}{\cos(\theta+\alpha)} = \dfrac{2}{\cos\theta}$;

$$\therefore\ \frac{4\cos\theta\cos\alpha}{\cos 2\theta + \cos 2\alpha} = \frac{2}{\cos\theta};$$

$$\therefore\ 2\cos^2\theta\cos\alpha = 2\cos^2\theta - 1 + 2\cos^2\alpha - 1;$$

$$\therefore\ \cos^2\theta(\cos\alpha - 1) = \cos^2\alpha - 1;$$

$$\therefore\ \cos^2\theta = \cos\alpha + 1 = 2\cos^2\frac{\alpha}{2};$$

whence $\cos\theta = \sqrt{2}\cos\dfrac{\alpha}{2}$.

15. We have $\sin\alpha\cos\alpha = \sin^2\beta$

$$= \frac{1}{2}(1 - \cos 2\beta);$$

$$\therefore\ \cos 2\beta = 1 - 2\sin\alpha\cos\alpha = (\cos\alpha - \sin\alpha)^2$$

$$= 2\cos^2\left(\frac{\pi}{4} + \alpha\right).$$

16. See figure of Art. 223.

$$OG = BG\cot BOG = BG\cot C = c\cos B\cot C$$
$$= 2R\cos B\cos C;$$

similarly $OH = 2R\cos C\cos A$, $OK = 2R\cos A\cos B$.

17. We have $\cos\theta = \dfrac{\cos u - e}{1 - e\cos u}$.

$$\therefore \frac{1-\cos\theta}{1+\cos\theta} = \frac{1 - e\cos u - \cos u + e}{1 - e\cos u + \cos u - e} = \frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)};$$

$$\therefore \tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{u}{2}.$$

18. The sum of the squares on the sides containing the right angle

$$= 4(1+\sin\theta)^2 + 4(1+\cos\theta)^2 + \cos^2\theta + \sin^2\theta + 4\cos\theta(1+\sin\theta) + 4\sin\theta(1+\cos\theta)$$

$$= 8 + 8(\sin\theta + \cos\theta) + 4 + 1 + 4(\sin\theta + \cos\theta) + 8\sin\theta\cos\theta$$

$$= 9 + 12(\sin\theta + \cos\theta) + 4(1 + 2\sin\theta\cos\theta)$$

$$= 9 + 12(\sin\theta + \cos\theta) + 4(\sin\theta + \cos\theta)^2$$

$$= \{3 + 2(\sin\theta + \cos\theta)\}^2.$$

$$\therefore \text{hypotenuse} = 3 + 2(\sin\theta + \cos\theta).$$

19. In the figure of Art. 227 let O be the centre of in-circle of $I_1I_2I_3$, and O_1 the centre of the ex-circle opposite to I_1. Let R' be the circum-radius of $I_1I_2I_3$.

Then, as in Art. 220, $OO_1 = 4R'\sin\dfrac{I_2I_1I_3}{2}$;

but $\angle I_2I_1I_3 = \dfrac{\pi}{2} - \dfrac{A}{2}$; and $R' = 2R$; [Arts. 221, 222];

$$\therefore OO_1 = 8R\sin\left(\frac{\pi}{4} - \frac{A}{4}\right) = 8R\sin\frac{B+C}{4}.$$

20. The sides of the ex-central triangle of the triangle $I_1I_2I_3$ are

$$4R'\cos\frac{\pi - A}{4}, \quad 4R'\cos\frac{\pi - B}{4}, \quad 4R'\cos\frac{\pi - C}{4}, \qquad \text{[Art. 221]},$$

that is, $8R\cos\dfrac{B+C}{4}$, $8R\cos\dfrac{C+A}{4}$, $8R\cos\dfrac{A+B}{4}$.

21. We have

$$(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) = (1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma);$$

$$\therefore \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = \pm\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2};$$

$$\therefore \text{each expression} = 8\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}$$

$$= \pm 8\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \cdot \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$= \pm\sin\alpha\sin\beta\sin\gamma.$$

22. Let $\alpha, \beta, \gamma, \delta$ be four angles such that $\alpha+\beta+\gamma+\delta=180°$;

then $\cos(\alpha+\beta)=\cos(180°-\overline{\gamma+\delta})=-\cos(\gamma+\delta)$;

$\therefore \cos\alpha\cos\beta+\cos\gamma\cos\delta=\sin\alpha\sin\beta+\sin\gamma\sin\delta$;

similarly $\cos\beta\cos\gamma+\cos\delta\cos\alpha=\sin\beta\sin\gamma+\sin\delta\sin\alpha$;

$\cos\gamma\cos\alpha+\cos\beta\cos\delta=\sin\gamma\sin\alpha+\sin\beta\sin\delta$;

$\therefore$ by addition we have the sum of the products of the cosines taken two together equal to the sum of the products of the sines taken two together.

23. (1) $II_1 . II_2 . II_3 = 4R\sin\frac{A}{2} . 4R\sin\frac{B}{2} . 4R\sin\frac{C}{2}$ [Art. 220]

$$=16R^2 . 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=16R^2 r.$$

(2) $II_1^2+I_2I_3^2=16R^2\sin^2\frac{A}{2}+16R^2\cos^2\frac{A}{2}$ [Arts. 220, 221]

$$=16R^2.$$

24. (1) Let α, β, γ be the angles;

then $\cos\alpha=\dfrac{\cos^2\frac{B}{2}+\cos^2\frac{C}{2}-\cos^2\frac{A}{2}}{2\cos\frac{B}{2}\cos\frac{C}{2}}=\dfrac{1+\cos\frac{B+C}{2}\cos\frac{B-C}{2}-\cos^2\frac{A}{2}}{2\cos\frac{B}{2}\cos\frac{C}{2}}$

$$=\frac{\sin\frac{A}{2}\cos\frac{B-C}{2}+\sin^2\frac{A}{2}}{2\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=\sin\frac{A}{2}=\cos\left(90°-\frac{A}{2}\right).$$

Thus the angles are $90°-\frac{A}{2}$, $90°-\frac{B}{2}$, $90°-\frac{C}{2}$.

(2) Here

$$\cos\alpha=\frac{\sin^2 2B+\sin^2 2C-\sin^2 2A}{2\sin 2B\sin 2C}=\frac{1-\cos 2(B+C)\cos 2(B-C)-\sin^2 2A}{2\sin 2B\sin 2C}$$

$$=\frac{-\cos 2A\cos 2(B-C)+\cos 2A}{2\sin 2B\sin 2C}$$

$$=\frac{-\cos 2A\{\cos 2(B-C)-\cos 2(B+C)\}}{2\sin 2B\sin 2C}=-\cos 2A.$$

Thus the angles are $180°-2A$, $180°-2B$, $180°-2C$.

25. The expression

$$= \{\sin(\theta+\alpha)+\sin(\theta+\beta)\}^2 - 2\sin(\theta+\alpha)\sin(\theta+\beta)$$
$$- 2\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta)$$

$$= \left\{2\sin\frac{2\theta+\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right\}^2 - 2\sin(\theta+\alpha)\sin(\theta+\beta)(1+\cos\overline{\alpha-\beta})$$

$$= 4\cos^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{2\theta+\alpha+\beta}{2} - \sin(\theta+\alpha)\sin(\theta+\beta)\right\}$$

$$= 2\cos^2\frac{\alpha-\beta}{2}\{1-\cos(2\theta+\alpha+\beta)-\cos(\alpha-\beta)+\cos(2\theta+\alpha+\beta)\}$$

$$= 2\cos^2\frac{\alpha-\beta}{2}(1-\cos\overline{\alpha-\beta}),$$

which is independent of θ.

26. See figure on page 220.

Since the quadrilateral is described about a circle,

$$\therefore\ a+c=b+d;\ \text{that is,}\ a-d=b-c.$$

Now $$a^2+d^2-2ad\cos A = BD^2 = b^2+c^2-2bc\cos C;$$

$$\therefore\ (a-d)^2+2ad(1-\cos A) = (b-c)^2+2bc(1-\cos C);$$

$$\therefore\ ad\sin^2\frac{A}{2} = bc\sin^2\frac{C}{2}.$$

27. Let the tangent parallel to BC meet AC in M; and let AG, the perpendicular from A to BC, meet the tangent in X; then

$$\frac{p}{a} = \frac{AM}{AC} = \frac{AX}{AG} = \frac{AG-2r}{AG}$$

$$= 1-\frac{ar}{\Delta} = 1-\frac{a}{s};$$

$$\therefore \frac{p}{a}+\frac{q}{b}+\frac{r}{c} = 1-\frac{a}{s}+1-\frac{b}{s}+1-\frac{c}{s}$$

$$= 3-\frac{2s}{s} = 1.$$

28. Take the figure of Art. 199. Then we have

$$\angle PAB = 13° 14' 12'', \quad \angle PBC = 56° 24' 36'', \quad \angle BPA = 43° 10' 24''.$$

Also $PC = 1566$. Let $AB = x$ ft.

$$x = \frac{PB \sin 43° 10' 24''}{\sin 13° 14' 12''}$$

$\log \sin 43° 10'$	$= \bar{1}\cdot 8351341$
$\frac{24}{60} \times 1347 =$	539
$\log PB$	$= 3\cdot 2741376$
	$3\cdot 1093256$
$\log \sin 13° 14' 12''$	$= \bar{1}\cdot 3597858$
$\log x$	$= 3\cdot 7495398$
$\log 5617\cdot 4$	$= 3\cdot 7495353$
	45
6	46

$$PB = 1566 \operatorname{cosec} 56° 24' 36''.$$

$\log \operatorname{cosec} 56° 24' =$	$\cdot 0793961$
subtract $\frac{36}{60} \times 839 =$	503
	$\cdot 0793458$
$\log 1566$	$= 3\cdot 1947918$
$\log PB$	$= 3\cdot 2741376$
$\log \sin 13° 14'$	$= \bar{1}\cdot 3596785$
$\frac{12}{60} \times 5369 =$	1073
	$\bar{1}\cdot 3597858$

Thus $x = 5617\cdot 46$ ft., whence it easily follows that the speed of the train is 21·3 miles per hour.

29. Let A represent the harbour, C the fort, B the position of the ship when 20 miles from C.

Then $AC = 27\cdot 23$ miles, $CB = 20$ miles, $\angle CAB = 46° 8' 8\cdot 6''$.

$$\sin B = \frac{27\cdot 23 \sin 46° 8' 8\cdot 6''}{20}.$$

$\log 27\cdot 23$	$= 1\cdot 4350476$
$\log \sin 46° 8'$	$= \bar{1}\cdot 8579078$
$\frac{86}{600} \times 1215 =$	174
	$\bar{1}\cdot 2929728$
$\log 20$	$= 1\cdot 3010300$
$\log \sin B$	$= \bar{1}\cdot 9919428$
$\log \sin 78° 59' =$	$\bar{1}\cdot 9919220$
	208

Diff. for $60'' = 246$;

$\therefore$ prop^{l}. increase $= \frac{208}{246} \times 60'' = 50\cdot 7''$.

$\therefore B = 78° 59' 50\cdot 7''$, or $101° 0' 9\cdot 3''$, both values being admissible since $a < b$.

Hence with the third figure of page 131 we have

$$\angle ACB_1 = 54° 52' 0\cdot 7'', \quad \angle ACB_2 = 32° 51' 42\cdot 1''.$$

In $\triangle ACB_1$,

$$AB_1 = \frac{20 \sin 54° 52' 0{\cdot}7''}{\sin 46° 8' 8{\cdot}6''},$$

$\log \sin 54° 52' = \bar{1}{\cdot}9126551$

$\frac{7}{600} \times 889 = 10$

$\log 20 = 1{\cdot}3010300$

$\bar{1}{\cdot}2136861$

$\log \sin 46° 8' 8{\cdot}6'' = \bar{1}{\cdot}8579252$

$\log AB_1 = \bar{1}{\cdot}3557609$

$\log 22{\cdot}686 = 1{\cdot}3557579$

30

$2 \quad 38$

$\therefore AB_1 = 22{\cdot}6862$ miles.

In $\triangle ACB_2$,

$$AB_2 = \frac{20 \sin 32° 51' 42{\cdot}1''}{\sin 46° 8' 8{\cdot}6''};$$

$\log \sin 32° 51' = \bar{1}{\cdot}7343529$

$\frac{421}{600} \times 1956 = 1372$

$\log 20 = 1{\cdot}3010300$

$\bar{1}{\cdot}0355201$

$\log \sin 46° 8' 8{\cdot}6'' = \bar{1}{\cdot}8579252$

$\log AB_2 = 1{\cdot}1775949$

$\log 15{\cdot}052 = 1{\cdot}1775942$

$\therefore AB_2 = 15{\cdot}052$ miles.

Thus the time taken is approximately 2·27 hours or 1·5 hours; that is the ship will be 20 miles from the fort in 2 hrs. 16 min. or in 1 hr. 30 min.

EXAMPLES. XIX. a. PAGE 235.

1. $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$;

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}$.

2. $\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$;

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{4}$.

3. $\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$;

$\therefore \theta = 2n\pi \pm \frac{\pi}{3}$.

4. $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$;

$\therefore \theta = n\pi + \frac{\pi}{3}$.

5. $\cot\theta = -\sqrt{3} = \cot\left(-\frac{\pi}{6}\right)$;

$\therefore \theta = n\pi - \frac{\pi}{6}$.

6. $\sec\theta = -\sqrt{2} = \sec\frac{3\pi}{4}$;

$\therefore \theta = 2n\pi \pm \frac{3\pi}{4}$.

7. $\cos^2\theta = \frac{1}{2}$;

$\therefore \cos\theta = \pm\frac{1}{\sqrt{2}}$;

$\theta = 2n\pi \pm \frac{\pi}{4}$, or $2n\pi \pm \left(\pi - \frac{\pi}{4}\right)$.

8. $\tan^2\theta = \frac{1}{3}$;

$\therefore \tan\theta = \pm\frac{1}{\sqrt{3}} = \tan\left(\pm\frac{\pi}{6}\right)$;

$\therefore \theta = n\pi \pm \frac{\pi}{6}$.

Both of these are included in $n\pi \pm \frac{\pi}{4}$.

9. $\operatorname{cosec}^2\theta=\frac{4}{3}$;

$\therefore \cot^2\theta=\frac{1}{3}$;

$\therefore \cot\theta=\pm\frac{1}{\sqrt{3}}=\cot\left(\pm\frac{\pi}{3}\right)$;

$\therefore \theta=n\pi\pm\frac{\pi}{3}$.

10. $\cos\theta=\cos\alpha$;

$\therefore \theta=2n\pi\pm\alpha$.

11. $\tan^2\theta=\tan^2\alpha$;

$\therefore \tan\theta=\pm\tan\alpha=\tan(\pm\alpha)$;

$\therefore \theta=n\pi\pm\alpha$.

12. $\sec^2\theta=\sec^2\alpha$;

$\therefore \tan^2\theta=\tan^2\alpha$;

$\therefore \theta=n\pi\pm\alpha$.

13. $\tan 2\theta=\tan\theta$;

$\therefore 2\theta=n\pi+\theta$;

$\therefore \theta=n\pi$.

14. $\operatorname{cosec}3\theta=\operatorname{cosec}3\alpha$;

$\therefore 3\theta=n\pi+(-1)^n 3\alpha$;

$\therefore \theta=\frac{n\pi}{3}+(-1)^n\alpha$.

15. $\cos 3\theta=\cos 2\theta$;

$\therefore 3\theta=2n\pi\pm 2\theta$;

$\therefore \theta=2n\pi$, or $\frac{2n\pi}{5}$.

16. $\sin 5\theta+\sin\theta=\sin 3\theta$;

$\therefore 2\sin 3\theta\cos 2\theta=\sin 3\theta$;

$\therefore \sin 3\theta=0$,

or $\cos 2\theta=\frac{1}{2}$,

whence $\theta=\frac{n\pi}{3}$, or $n\pi\pm\frac{\pi}{6}$.

17. $\cos\theta-\cos 7\theta=\sin 4\theta$;

$\therefore 2\sin 4\theta\sin 3\theta=\sin 4\theta$;

$\therefore \sin 4\theta=0$, or $\sin 3\theta=\frac{1}{2}$;

whence $\theta=\frac{n\pi}{4}$, or $\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$.

18. $\sin 4\theta+\sin 2\theta-(\sin 3\theta+\sin\theta)=0$;

$\therefore 2\sin 3\theta\cos\theta-2\sin 2\theta\cos\theta=0$;

$\therefore 2\cos\theta\,.\,2\cos\frac{5\theta}{2}\sin\frac{\theta}{2}=0$;

$\therefore \cos\theta=0$, or $\cos\frac{5\theta}{2}=0$, or $\sin\frac{\theta}{2}=0$.

$\therefore \theta=\frac{(2n+1)\pi}{2}$, or $\frac{(2n+1)\pi}{5}$, or $2n\pi$.

19. As in Example 18, we obtain

$$4\cos\theta\cos 4\theta\cos 2\theta=0;$$

whence $\theta=\frac{(2n+1)\pi}{2}$, or $\frac{(2n+1)\pi}{4}$, or $\frac{(2n+1)\pi}{8}$.

20. $\sin 5\theta \cos\theta = \sin 6\theta \cos 2\theta$;

$\therefore \sin 6\theta + \sin 4\theta = \sin 8\theta + \sin 4\theta$;

$\therefore \sin 8\theta - \sin 6\theta = 0$;

$\therefore 2\sin\theta \cos 7\theta = 0$;

$\therefore \sin\theta = 0$, or $\cos 7\theta = 0$;

whence $\theta = n\pi$, or $\frac{(2n+1)\pi}{14}$.

21. $\sin 11\theta \sin 4\theta + \sin 5\theta \sin 2\theta = 0$;

$\therefore \cos 7\theta - \cos 15\theta + \cos 3\theta - \cos 7\theta = 0$;

$\therefore 2\sin 9\theta \sin 6\theta = 0$;

whence $\theta = \frac{n\pi}{9}$, or $\frac{n\pi}{6}$.

22. $\sqrt{2}\cos 3\theta - \cos\theta = \cos 5\theta$;

$\therefore \sqrt{2}\cos 3\theta = 2\cos 3\theta \cos 2\theta$;

$\therefore \cos 3\theta = 0$, or $\cos 2\theta = \frac{1}{\sqrt{2}}$;

$\therefore \theta = \frac{(2n+1)\pi}{6}$, or $n\pi \pm \frac{\pi}{8}$.

23. $\sin 7\theta - \sqrt{3}\cos 4\theta = \sin\theta$;

$\therefore 2\cos 4\theta \sin 3\theta = \sqrt{3}\cos 4\theta$;

$\therefore \cos 4\theta = 0$, or $\sin 3\theta = \frac{\sqrt{3}}{2}$;

whence $\theta = \frac{(2n+1)\pi}{8}$, or $\frac{n\pi}{3} + (-1)^n \frac{\pi}{9}$.

24. $1 + \cos\theta = 2\sin^2\theta$;

$\therefore 1 + \cos\theta = 2 - 2\cos^2\theta$;

$\therefore 2\cos^2\theta + \cos\theta - 1 = 0$;

$\therefore \cos\theta = -1$, or $\frac{1}{2}$;

$\therefore \theta = (2n+1)\pi$, or $2n\pi \pm \frac{\pi}{3}$.

25. $\tan^2\theta + \sec\theta = 1$;

$\therefore \sec^2\theta + \sec\theta - 2 = 0$;

$\therefore \sec\theta = -2$, or 1;

$\therefore \theta = 2n\pi \pm \frac{2\pi}{3}$, or $2n\pi$.

26. $\cot^2\theta - 1 = \operatorname{cosec}\theta;$

$\therefore \operatorname{cosec}^2\theta - \operatorname{cosec}\theta - 2 = 0;$

$\therefore \operatorname{cosec}\theta = 2, \text{ or } -1;$

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}, \text{ or } n\pi + (-1)^n \frac{3\pi}{2}.$

27. $\cot\theta - \tan\theta = 2;$

$\therefore \cot^2\theta - 1 = 2\cot\theta;$

$\therefore \cot 2\theta = 1;$

$\therefore \theta = \frac{n\pi}{2} + \frac{\pi}{8}.$

28. $2\cos\theta = -1; \ \therefore \theta = 2n\pi \pm \frac{2\pi}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$

$2\sin\theta = \sqrt{3}; \ \therefore \theta = n\pi + (-1)^n \frac{2\pi}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$

From (1) we see that the multiple of π must be even, and from (2) that the sign before the second term must be positive when the multiple of π is even;

$\therefore \theta = 2n\pi + \frac{2\pi}{3}.$

29. $\sec\theta = \sqrt{2}; \ \therefore \theta = 2n\pi \pm \frac{\pi}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$

$\tan\theta = -1; \ \therefore \theta = n\pi - \frac{\pi}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$

From (1) we see that the multiple of π must be even, and from (2), that the sign before the second term must be negative;

$\therefore \theta = 2n\pi - \frac{\pi}{4}.$

EXAMPLES. XIX. b. PAGE 237.

1. $\tan p\theta = \cot q\theta;$

$\therefore \tan p\theta = \tan\left(\frac{\pi}{2} - q\theta\right).$

$\therefore p\theta = n\pi + \frac{\pi}{2} - q\theta;$

$\therefore \theta = \frac{(2n+1)\pi}{2(p+q)}.$

2. $\sin m\theta + \cos n\theta = 0;$

$$\therefore \cos n\theta = \cos\left(\frac{\pi}{2} + m\theta\right);$$

$$\therefore n\theta = 2\kappa\pi \pm \left(\frac{\pi}{2} + m\theta\right);$$

$$\therefore \theta = \frac{(4\kappa+1)\pi}{2(n-m)}, \text{ or } \frac{(4\kappa-1)\pi}{2(n+m)}.$$

3. $\cos\theta - \sqrt{3}\sin\theta = 1;$

$$\therefore \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = \frac{1}{2};$$

$$\therefore \cos\left(\theta + \frac{\pi}{3}\right) = \cos\frac{\pi}{3};$$

$$\therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3};$$

$$\therefore \theta = 2n\pi, \text{ or } 2n\pi - \frac{2\pi}{3}.$$

4. $\sin\theta - \sqrt{3}\cos\theta = 1;$

$$\therefore \frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2};$$

$$\therefore \cos\left(\theta + \frac{\pi}{6}\right) = \cos\frac{2\pi}{3};$$

$$\therefore \theta + \frac{\pi}{6} = 2n\pi \pm \frac{2\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi - \frac{5\pi}{6};$$

that is,

$$\theta = 2n\pi + \frac{\pi}{2}, \text{ or } (2n+1)\pi + \frac{\pi}{6}.$$

5. $\cos\theta = \sqrt{3}(1 - \sin\theta);$

$$\therefore \frac{1}{2}\cos\vartheta + \frac{\sqrt{3}}{2}\sin\theta = \frac{\sqrt{3}}{2};$$

$$\therefore \cos\left(\theta - \frac{\pi}{3}\right) = \cos\frac{\pi}{6};$$

$$\therefore \theta - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{2}, \text{ or } 2n\pi + \frac{\pi}{6}.$$

6. $\sin\theta+\sqrt{3}\cos\theta=\sqrt{2}$;

$\therefore\ \cos\left(\theta-\frac{\pi}{6}\right)=\cos\frac{\pi}{4}$;

$\therefore\ \theta-\frac{\pi}{6}=2n\pi\pm\frac{\pi}{4}$;

$\therefore\ \theta=2n\pi+\frac{5\pi}{12}$, or $2n\pi-\frac{\pi}{12}$.

7. $\cos\theta-\sin\theta=\frac{1}{\sqrt{2}}$;

$\therefore\ \frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta=\frac{1}{2}$,

$\therefore\ \cos\left(\theta+\frac{\pi}{4}\right)=\cos\frac{\pi}{3}$;

$\therefore\ \theta+\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}$;

$\therefore\ \theta=2n\pi+\frac{\pi}{12}$, or $2n\pi-\frac{7\pi}{12}$.

8. $\cos\theta+\sin\theta+\sqrt{2}=0$;

$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=-1$;

$\therefore\ \theta-\frac{\pi}{4}=2n\pi\pm\pi$;

$\therefore\ \theta=2n\pi+\frac{5\pi}{4}$, or $2n\pi-\frac{3\pi}{4}$.

9. $\operatorname{cosec}\theta+\cot\theta=\sqrt{3}$;

$\therefore\ 1+\cos\theta=\sqrt{3}\sin\theta$;

$\therefore\ \frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta=-\frac{1}{2}$;

$\therefore\ \cos\left(\theta+\frac{\pi}{3}\right)=-\frac{1}{2}$;

$\therefore\ \theta+\frac{\pi}{3}=2n\pi\pm\frac{2\pi}{3}$;

whence $\theta=2n\pi+\frac{\pi}{3}$, or $(2n-1)\pi$;

which may be written $\theta=2n\pi+\frac{\pi}{3}$, or $(2n+1)\pi$.

10. $\cot\theta-\cot 2\theta=2$;

$\therefore\ \operatorname{cosec} 2\theta=2$;

$\therefore\ \theta=\frac{n\pi}{2}+(-1)^n\frac{\pi}{12}$.

11. $2\sin\theta\sin 3\theta = 1;$

$\therefore \cos 2\theta - \cos 4\theta = 1;$

$\therefore \cos 2\theta = 2\cos^2 2\theta;$

$\therefore \cos 2\theta = 0, \text{ or } \frac{1}{2};$

$\therefore \theta = \frac{(2n+1)\pi}{4}, \text{ or } n\pi \pm \frac{\pi}{6}.$

12. $\sin 3\theta = 8\sin^3\theta;$

$\therefore 3\sin\theta - 4\sin^3\theta = 8\sin^3\theta;$

$\therefore \sin\theta = 4\sin^3\theta;$

$\therefore \sin\theta = 0, \text{ or } \pm\frac{1}{2};$

$\therefore \theta = n\pi, \text{ or } n\pi \pm \frac{\pi}{6}.$

13. $\tan\theta + \tan 3\theta = 2\tan 2\theta;$

$\therefore \frac{\sin 4\theta}{\cos 3\theta\cos\theta} = \frac{2\sin 2\theta}{\cos 2\theta};$

$\therefore \sin 2\theta = 0;$ whence $\theta = \frac{n\pi}{2},$

or $\cos^2 2\theta = \cos\theta\cos 3\theta;$

$\therefore 2\cos^2 2\theta = \cos 4\theta + \cos 2\theta = 2\cos^2 2\theta - 1 + \cos 2\theta;$

$\therefore \cos 2\theta = 1;$

whence $\theta = n\pi;$

$\therefore$ all the values are included in $\theta = \frac{n\pi}{2}.$

14. $\cos\theta - \sin\theta = \cos 2\theta;$

$\therefore 2\sin\frac{3\theta}{2}\sin\frac{\theta}{2} = \sin\theta;$

$\therefore \sin\frac{\theta}{2} = 0,$ whence $\theta = 2n\pi,$

$\sin\frac{3\theta}{2} = \cos\frac{\theta}{2};$

$\therefore \frac{\theta}{2} = 2n\pi \pm \left(\frac{\pi}{2} - \frac{3\theta}{2}\right);$

$\therefore \theta = n\pi + \frac{\pi}{4}, \text{ or } -\theta = 2n\pi - \frac{\pi}{2};$

$\therefore$ the values of θ may be written $2n\pi,\ n\pi + \frac{\pi}{4},\ 2n\pi + \frac{\pi}{2}.$

15. $$\operatorname{cosec}\theta+\sec\theta=2\sqrt{2};$$
$$\therefore\ \sin\theta+\cos\theta=2\sqrt{2}\sin\theta\cos\theta;$$
$$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=2\sin\theta\cos\theta=\sin 2\theta;$$
$$\therefore\ \theta-\frac{\pi}{4}=2n\pi\pm\left(\frac{\pi}{2}-2\theta\right);$$
whence $$\theta=\frac{2n\pi}{3}+\frac{\pi}{4},\ \text{or}\ 2n\pi+\frac{\pi}{4}.$$

16. $$\sec\theta-\operatorname{cosec}\theta=2\sqrt{2};$$
$$\therefore\ 1-\sin 2\theta=8\sin^2\theta\cos^2\theta=2\sin^2 2\theta;$$
$$\therefore\ \sin 2\theta=-1,\ \text{or}\ \frac{1}{2};$$
$$\therefore\ \theta=n\pi-\frac{\pi}{4},\ \text{or}\ \frac{n\pi}{2}+(-1)^n\frac{\pi}{12}.$$

The equation may also be solved in the same way as Ex. 15.

17. $$\frac{1}{\cos 4\theta}-\frac{1}{\cos 2\theta}=2;$$
$$\cos 2\theta-\cos 4\theta=2\cos 4\theta\cos 2\theta$$
$$=\cos 6\theta+\cos 2\theta;$$
$$\therefore\ \cos 6\theta+\cos 4\theta=0,$$
$$2\cos 5\theta\cos\theta=0;$$
$$\therefore\ \theta=(2n+1)\frac{\pi}{2},\ \text{or}\ 5\theta=(2n+1)\frac{\pi}{2}.$$

18. $$\cos 3\theta+8\cos^3\theta=0;$$
$$\therefore\ 4\cos^3\theta=\cos\theta;$$
$$\therefore\ \cos\theta=0,\ \text{or}\ \pm\frac{1}{2};$$
$$\therefore\ \theta=\frac{(2n+1)\pi}{2},\ 2n\pi\pm\frac{\pi}{3},\ 2n\pi\pm\left(\pi-\frac{\pi}{3}\right);$$
that is, the values of θ are $$\frac{(2n+1)\pi}{2},\ n\pi\pm\frac{\pi}{3}.$$

19. $$1+\sqrt{3}\tan^2\theta=(1+\sqrt{3})\tan\theta;$$
$$\therefore\ (\sqrt{3}\tan\theta-1)(\tan\theta-1)=0;$$
$$\therefore\ \tan\theta=1,\ \text{or}\ \frac{1}{\sqrt{3}};$$
$$\therefore\ \theta=n\pi+\frac{\pi}{4},\ \text{or}\ n\pi+\frac{\pi}{6}.$$

20. $\tan^3\theta+\cot^3\theta=8\operatorname{cosec}^3 2\theta+12$;

$\therefore \sin^6\theta+\cos^6\theta=1+12\sin^3\theta\cos^3\theta$;

$\therefore \sin^4\theta-\sin^2\theta\cos^2\theta+\cos^4\theta=1+12\sin^3\theta\cos^3\theta$;

$\therefore 1-3\sin^2\theta\cos^2\theta=1+12\sin^3\theta\cos^3\theta$;

$\therefore \sin^3\theta\cos^2\theta(4\sin\theta\cos\theta+1)=0$;

whence $\sin 2\theta=0$, or $-\frac{1}{2}$;

$\therefore \theta=\frac{n\pi}{2}$, or $\frac{n\pi}{2}+(-1)^{n+1}\frac{\pi}{12}$.

21. $\sin\theta=\sqrt{2}\sin\phi$, $\sqrt{3}\cos\theta=\sqrt{2}\cos\phi$;

$\therefore$ by squaring and adding we have $\sin^2\theta+3\cos^2\theta=2$,

that is, $1+2\cos^2\theta=2$, whence $\cos\theta=\pm\frac{1}{\sqrt{2}}$;

$\therefore \theta=2n\pi\pm\frac{\pi}{4}$, or $2n\pi\pm\left(\pi-\frac{\pi}{4}\right)$;

both of which are included in $\theta=n\pi\pm\frac{\pi}{4}$.

Again, we have $\cos\phi=\frac{\sqrt{3}}{\sqrt{2}}\cos\theta=\pm\frac{\sqrt{3}}{2}$;

$\therefore \phi=2n\pi\pm\frac{\pi}{6}$, or $2n\pi\pm\left(\pi-\frac{\pi}{6}\right)$,

which are both included in $\phi=n\pi\pm\frac{\pi}{6}$.

22. $\operatorname{cosec}\theta=\sqrt{3}\operatorname{cosec}\phi$, $\cot\theta=3\cot\phi$.

By squaring and subtracting we have

$1=3(\operatorname{cosec}^2\phi-3\cot^2\phi)=3(1-2\cot^2\phi)$;

$\therefore \cot\phi=\pm\frac{1}{\sqrt{3}}$; whence $\phi=n\pi\pm\frac{\pi}{3}$.

Also $\cot\theta=\pm\sqrt{3}$; whence $\theta=n\pi\pm\frac{\pi}{6}$.

23. $\sec\phi=\sqrt{2}\sec\theta$, $\tan\phi=\sqrt{3}\tan\theta$.

Subtracting the square of the second equation from the square of the first, we obtain

$1=2\sec^2\theta-3\tan^2\theta=2-\tan^2\theta$;

$\therefore \tan\theta=\pm1$; whence $\theta=n\pi\pm\frac{\pi}{4}$.

Also $\tan\phi=\pm\sqrt{3}$; whence $\phi=n\pi\pm\frac{\pi}{3}$.

24. If $$\theta+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{6};$$

then $$\sin\left(\theta+\frac{\pi}{4}\right)=\sin\frac{\pi}{6};$$

$$\therefore\ \cos\left(\frac{\pi}{2}-\theta-\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right);$$

$$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos\frac{\pi}{3};$$

$$\therefore\ \theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{3}.$$

Which proves that the same series of angles are given by the two equations.

25. In the figure on page 252 let OE be drawn bisecting the angle P_1OP_2, and let the angles P_1OE, P_2OE each be equal to α;

then the formula $\left(2n+\frac{1}{4}\right)\pi\pm\alpha$ comprises angles whose boundary lines are OP_1 or OP_2.

Again in the formula $\left(n-\frac{1}{4}\right)\pi+(-1)^n\left(\frac{\pi}{2}-\alpha\right)$ the terms $n\pi+(-1)^n\frac{\pi}{2}$ comprise angles whose boundary lines are OY, whether n be odd or even.

Hence the whole formula comprises angles formed by starting again from OY and turning through an angle $-\frac{\pi}{4}-(-1)^n\alpha$; that is, $-\frac{\pi}{4}\pm\alpha$.

Hence the second formula also comprises angles whose boundary lines are OP_1 or OP_2.

EXAMPLES. XIX. c. PAGE 242.

1. Let $$\theta=\sin^{-1}\frac{12}{13};\ \text{then}\ \operatorname{cosec}\theta=\frac{13}{12}.$$

$$\therefore\ \cot^2\theta=\frac{169}{144}-1=\frac{25}{144};$$

$$\therefore\ \theta=\cot^{-1}\frac{5}{12}.$$

2. Let $\theta=\operatorname{cosec}^{-1}\frac{17}{8}$; then $\operatorname{cosec}\theta=\frac{17}{8}$, and $\cot\theta=\frac{15}{8}$.

$$\therefore\ \theta=\tan^{-1}\frac{8}{15}.$$

3. Let $$\theta=\tan^{-1}x;\ \text{then}\ \tan\theta=x;$$

$$\therefore\ \sec^2\theta=1+\tan^2\theta=1+x^2,$$

that is, $$\sec(\tan^{-1}x)=\sqrt{1+x^2}.$$

4. $$2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}=\tan^{-1}\frac{3}{4}.$$

5. $$\tan^{-1}\frac{4}{3}-\tan^{-1}1=\tan^{-1}\frac{\frac{4}{3}-1}{1+\frac{4}{3}}=\tan^{-1}\frac{1}{7}.$$

6. $$\tan^{-1}\frac{2}{11}+\cot^{-1}\frac{24}{7}=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$$
$$=\tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{7}{132}}=\tan^{-1}\frac{1}{2}.$$

7. $$\cot^{-1}\frac{4}{3}-\cot^{-1}\frac{15}{8}=\cot^{-1}\frac{\frac{4}{3}\cdot\frac{15}{8}+1}{\frac{15}{8}-\frac{4}{3}}=\cot^{-1}\frac{84}{13}.$$

8. $$2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}+\tan^{-1}\frac{1}{4}$$
$$=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{1}{4}$$
$$=\tan^{-1}\frac{\frac{5}{12}+\frac{1}{4}}{1-\frac{5}{48}}=\tan^{-1}\frac{32}{43}.$$

9. $$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\tan^{-1}1.$$

Again, $$\tan^{-1}\frac{5}{6}+\tan^{-1}\frac{1}{11}=\tan^{-1}\frac{\frac{5}{6}+\frac{1}{11}}{1-\frac{5}{66}}=\tan^{-1}1.$$

10. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{18}=\tan^{-1}\dfrac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}+\tan^{-1}\frac{1}{18}$

$$=\tan^{-1}\frac{3}{11}+\tan^{-1}\frac{1}{18}=\tan^{-1}\dfrac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{198}}$$

$$=\tan^{-1}\frac{1}{3}=\cot^{-1}3.$$

11. $\tan^{-1}\frac{3}{5}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{5}+\tan^{-1}\frac{3}{4}$

$$=\tan^{-1}\dfrac{\frac{3}{5}+\frac{3}{4}}{1-\frac{9}{20}}=\tan^{-1}\frac{27}{11}.$$

12. $2\cot^{-1}\frac{5}{4}=2\tan^{-1}\frac{4}{5}=\tan^{-1}\dfrac{\frac{8}{5}}{1-\frac{16}{25}}=\tan^{-1}\frac{40}{9}.$

13. $2\tan^{-1}\frac{8}{15}=\tan^{-1}\dfrac{\frac{16}{15}}{1-\frac{64}{225}}=\tan^{-1}\frac{240}{161}=\sin^{-1}\frac{240}{\sqrt{161^2+240^2}}$

$$=\sin^{-1}\frac{240}{289}.$$

14. Let $\sin^{-1}x=\theta$; then $\sin\theta=x$, $\cos\theta=\sqrt{1-x^2}$.

$\therefore\ \sin(2\sin^{-1}x)=\sin 2\theta=2\sin\theta\cos\theta=2x\sqrt{1-x^2}$.

15. Let $\sin^{-1}\sqrt{\frac{1-x}{2}}=\theta$; then $\sin\theta=\sqrt{\frac{1-x}{2}}$.

Now $\cos 2\theta=1-(1-x)=x$; whence $2\theta=\cos^{-1}x$.

That is, $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}$.

16. Let $\tan^{-1}\sqrt{\frac{x}{a}}=\theta$; then $\tan\theta=\sqrt{\frac{x}{a}}$.

Now $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{a-x}{a+x}$; whence $2\theta=\cos^{-1}\frac{a-x}{a+x}$.

That is, $2\tan^{-1}\sqrt{\frac{x}{a}}=\cos^{-1}\frac{a-x}{a+x}$.

17. $2\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{5}=2\tan^{-1}\frac{\frac{1}{8}+\frac{1}{5}}{1-\frac{1}{40}}+\tan^{-1}\frac{1}{7}$

$=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}$

$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{28}}=\tan^{-1}1=\frac{\pi}{4}$.

18. Let $\theta=\sin^{-1}a,\ \phi=\cos^{-1}b$;

then $\sin\theta=a,\ \cos\theta=\sqrt{1-a^2},\ \cos\phi=b,\ \sin\phi=\sqrt{1-b^2}$;

$\therefore\ \cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi=b\sqrt{1-a^2}+a\sqrt{1-b^2}$;

$\therefore\ \sin^{-1}a-\cos^{-1}b=\theta-\phi=\cos^{-1}\{b\sqrt{1-a^2}+a\sqrt{1-b^2}\}$.

[In some of the examples which follow diagrams may be used with advantage as in Examples 2 and 3 of Art. 249.]

19. $\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{\sqrt{5}}=\cot^{-1}\frac{3}{4}+\cot^{-1}2=\cot^{-1}\frac{\frac{3}{2}-1}{\frac{3}{4}+2}=\cot^{-1}\frac{2}{11}$.

20. $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\cos^{-1}\frac{63}{65}+\tan^{-1}\frac{5}{12}$

$=\tan^{-1}\frac{\sqrt{65^2-63^2}}{63}+\tan^{-1}\frac{5}{12}$

$=\tan^{-1}\frac{16}{63}+\tan^{-1}\frac{5}{12}$

$=\tan^{-1}\frac{192+315}{756-80}=\tan^{-1}\frac{507}{676}=\tan^{-1}\frac{3}{4}$

$=\sin^{-1}\frac{3}{5}$.

21. $\tan^{-1} m + \tan^{-1} n = \tan^{-1} \dfrac{m+n}{1-mn}$

$$= \cos^{-1} \frac{1-mn}{\sqrt{(m+n)^2+(1-mn)^2}} = \cos^{-1} \frac{1-mn}{\sqrt{1+m^2+n^2+m^2n^2}}$$

$$= \cos^{-1} \frac{1-mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

22. Let $\cos^{-1} \dfrac{20}{29} = \theta, \quad \tan^{-1} \dfrac{16}{63} = \phi;$

then we have $\sin\theta = \dfrac{21}{29}, \quad \sin\phi = \dfrac{16}{65}, \quad \cos\phi = \dfrac{63}{65};$

$$\therefore \cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi = \frac{20}{29}\cdot\frac{63}{65} + \frac{21}{29}\cdot\frac{16}{65} = \frac{1596}{1885};$$

that is, $$\cos^{-1}\frac{20}{29} - \tan^{-1}\frac{16}{63} = \theta - \phi = \cos^{-1}\frac{1596}{1885}.$$

23. Let $\cos^{-1}\sqrt{\dfrac{2}{3}} = \theta$, then $\cos\theta = \sqrt{\dfrac{2}{3}}, \quad \sin\theta = \dfrac{1}{\sqrt{3}}.$

Now $$\cos\left(\theta - \frac{\pi}{6}\right) = \cos\theta\cos\frac{\pi}{6} + \sin\theta\sin\frac{\pi}{6}$$

$$= \sqrt{\frac{2}{3}\cdot\frac{3}{4}} + \sqrt{\frac{1}{3}\cdot\frac{1}{4}}$$

$$= \frac{\sqrt{6}+1}{2\sqrt{3}}.$$

$$\therefore \theta - \frac{\pi}{6} = \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}};$$

that is, $$\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}} = \frac{\pi}{6}.$$

24. Second side $= 2\cdot\dfrac{x+x^3}{1-x^4} = \dfrac{2x}{1-x^2}$

$$= \tan\left(\tan^{-1}\frac{2x}{1-x^2}\right)$$

$$= \tan(2\tan^{-1}x).$$

25. Second side $= \tan^{-1}a - \tan^{-1}b + \tan^{-1}b - \tan^{-1}c + \tan^{-1}c$

$$= \tan^{-1}a.$$

26. Let $\tan^{-1}x = \alpha, \quad \tan^{-1}y = \beta, \quad \tan^{-1}z = \gamma;$

then $$\alpha + \beta + \gamma = \pi;$$

$$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma; \text{ [Art. 135, Ex. 2.]}$$

that is, $$x + y + z = xyz.$$

27. We have $u=\cot^{-1}\sqrt{\cos\alpha}-\cot^{-1}\sqrt{\dfrac{1}{\cos\alpha}}$

$$=\cot^{-1}\frac{1+1}{\sqrt{\dfrac{1}{\cos\alpha}}-\sqrt{\cos\alpha}}=\cot^{-1}\frac{2\sqrt{\cos\alpha}}{1-\cos\alpha};$$

$$\therefore\ \cot^2 u=\frac{4\cos\alpha}{(1-\cos\alpha)^2};$$

$$\therefore\ \operatorname{cosec}^2 u=1+\cot^2 u=\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)^2;$$

$$\therefore\ \sin u=\frac{1-\cos\alpha}{1+\cos\alpha}=\tan^2\frac{\alpha}{2}.$$

EXAMPLES. XIX. d. PAGE 244.

1. $\sin^{-1}x=\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}$;

$\therefore\ x=\sqrt{1-x^2}$; whence $x=\pm\dfrac{1}{\sqrt{2}}$.

2. $\tan^{-1}x=\cot^{-1}x=\tan^{-1}\dfrac{1}{x}$;

$\therefore\ x=\dfrac{1}{x}$; whence $x=\pm 1$.

3. $\tan^{-1}(x+1)-\tan^{-1}(x-1)=\cot^{-1}2$;

$\therefore\ \tan^{-1}\dfrac{2}{x^2}=\tan^{-1}\dfrac{1}{2}$; whence $x=\pm 2$.

4. $\cot^{-1}x+\cot^{-1}2x=\dfrac{3\pi}{4}$;

$\therefore\ \cot^{-1}\dfrac{2x^2-1}{3x}=\dfrac{3\pi}{4}$;

$\therefore\ \dfrac{2x^2-1}{3x}=\cot\dfrac{3\pi}{4}=-1$;

$\therefore\ 2x^2+3x-1=0$; whence $x=\dfrac{-3\pm\sqrt{17}}{4}$.

5. $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(3x-2)$;

$\therefore\ \sin(\sin^{-1}x-\cos^{-1}x)=3x-2$;

$\therefore\ x^2-\sqrt{1-x^2}.\sqrt{1-x^2}=3x-2$;

that is, $2x^2-1=3x-2$;

$\therefore\ 2x^2-3x+1=0$; whence $x=1$, or $\dfrac{1}{2}$.

6. $\cos^{-1}x-\sin^{-1}x=\cos^{-1}x\sqrt{3}$;

$\therefore\ \cos(\cos^{-1}x-\sin^{-1}x)=x\sqrt{3}$;

$\therefore\ x\sqrt{1-x^2}+x\sqrt{1-x^2}=x\sqrt{3}$;

that is, $2x\sqrt{1-x^2}=x\sqrt{3}$; whence $x=0$, or $\pm\frac{1}{2}$.

7. $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$;

$\therefore\ \tan^{-1}\dfrac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{x^2-1}{x^2-4}}=\frac{\pi}{4}$;

$\therefore\ \frac{2x^2-4}{-3}=1$;

$\therefore\ 2x^2=1$, or $x=\pm\frac{1}{\sqrt{2}}$.

8. $2\cot^{-1}2+\cos^{-1}\frac{3}{5}=\operatorname{cosec}^{-1}x$;

$\therefore\ \cot^{-1}\frac{3}{4}+\cot^{-1}\frac{3}{4}=\operatorname{cosec}^{-1}x$;

$\therefore\ \cot^{-1}\dfrac{\frac{9}{16}-1}{\frac{3}{2}}=\operatorname{cosec}^{-1}x$;

$\therefore\ \cot^{-1}-\frac{7}{24}=\operatorname{cosec}^{-1}x=\cot^{-1}\sqrt{x^2-1}$;

$\therefore\ \frac{49}{24^2}=x^2-1$; whence $x=\pm\frac{25}{24}$.

9. $\tan^{-1}x+\tan^{-1}(1-x)=2\tan^{-1}\sqrt{x-x^2}$;

$\therefore\ \tan^{-1}\frac{1}{1-x+x^2}=\tan^{-1}\frac{2\sqrt{x-x^2}}{1-x+x^2}$;

$\therefore\ 1=4(x-x^2)$; whence $x=\frac{1}{2}$.

10. $\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}=2\tan^{-1}x$;

$\therefore\ \tan^{-1}\frac{2a}{1-a^2}-\tan^{-1}\frac{2b}{1-b^2}=2\tan^{-1}x$;

$\therefore\ 2\tan^{-1}a-2\tan^{-1}b=2\tan^{-1}x$;

$\therefore\ \tan^{-1}a-\tan^{-1}b=\tan^{-1}x$; whence $x=\frac{a-b}{1+ab}$.

11. $$\sin^{-1}\frac{2a}{1+a^2}+\tan^{-1}\frac{2x}{1-x^2}=\cos^{-1}\frac{1-b^2}{1+b^2};$$

$$\therefore\ \tan^{-1}\frac{2a}{1-a^2}+\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{2b}{1-b^2};$$

$$\therefore\ 2\tan^{-1}a+2\tan^{-1}x=2\tan^{-1}b;\ \text{whence}\ x=\frac{b-a}{1+ab}.$$

12. $$\cot^{-1}\frac{x^2-1}{2x}+\tan^{-1}\frac{2x}{x^2-1}+\frac{4\pi}{3}=0;$$

$$\therefore\ 2\tan^{-1}\frac{2x}{x^2-1}+\frac{4\pi}{3}=0;$$

$$\therefore\ \tan^{-1}\frac{2x}{1-x^2}=\frac{2\pi}{3};$$

$$\tan^{-1}x=\frac{\pi}{3};\ \text{whence}\ x=\sqrt{3}.$$

13. $$\sin^{-1}\frac{2ab}{a^2+b^2}=\sin^{-1}\frac{\frac{2b}{a}}{1+\frac{b^2}{a^2}}=\tan^{-1}\frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}};$$

$$\therefore\ \sin^{-1}\frac{2ab}{a^2+b^2}+\sin^{-1}\frac{2cd}{c^2+d^2}=2\tan^{-1}\frac{b}{a}+2\tan^{-1}\frac{d}{c}$$

$$=2\tan^{-1}\frac{\frac{b}{a}+\frac{d}{c}}{1-\frac{bd}{ac}}$$

$$=2\tan^{-1}\frac{bc+ad}{ac-bd}$$

$$=2\tan^{-1}\frac{y}{x}=\sin^{-1}\frac{2xy}{x^2+y^2},$$

where $y=bc+ad,\ x=ac-bd.$

14. $$\sin\left[2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right]=0;$$

$$\therefore\ \sin\left[2\cos^{-1}\left\{\cot\left(\tan^{-1}\frac{2x}{1-x^2}\right)\right\}\right]=0;$$

$$\therefore\ \sin\left[2\cos^{-1}\frac{1-x^2}{2x}\right]=0;$$

$$\therefore\ \frac{1-x^2}{2x}\cdot\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}=0;$$

whence $x=\pm1$, or $1-x^2=\pm2x$;

that is, $x=\pm1$, or $\pm(1\pm\sqrt{2})$.

15. $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$;

$$\therefore\ \frac{2\cos\theta}{1-\cos^2\theta}=2\operatorname{cosec}\theta;$$

$$\therefore\ \frac{2\cos\theta}{\sin^2\theta}=\frac{2}{\sin\theta};$$

$$\therefore\ \cot\theta=1;\ \text{whence}\ \theta=n\pi+\frac{\pi}{4}.$$

16. $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$;

$$\therefore\ \pi\sin\theta=\frac{\pi}{2}\pm\pi\cos\theta;$$

$$\therefore\ \sin\theta\pm\cos\theta=\frac{1}{2};$$

$$\therefore\ 1\pm2\sin\theta\cos\theta=\frac{1}{4};$$

$$\therefore\ \sin 2\theta=\pm\frac{3}{4};$$

$$\therefore\ 2\theta=\pm\sin^{-1}\frac{3}{4}.$$

17. $\sin(\pi\cot\theta)=\cos(\pi\tan\theta)$;

$$\therefore\ \pi\tan\theta=2n\pi\pm\left(\frac{\pi}{2}-\pi\cot\theta\right);$$

$$\therefore\ \tan\theta\pm\cot\theta=\frac{4n\pm1}{2};$$

$$\therefore\ \frac{\sin^2\theta\pm\cos^2\theta}{\sin 2\theta}=\frac{4n\pm1}{4};$$

that is, either $\cot 2\theta$ or $\operatorname{cosec} 2\theta$ is of the form $\frac{4n+1}{4}$.

18. $\tan(\pi\cot\theta)=\cot(\pi\tan\theta)$;

$$\therefore\ \pi\tan\theta=n\pi+\frac{\pi}{2}-\pi\cot\theta;$$

$$\therefore\ \tan^2\theta-\frac{2n+1}{2}\tan\theta+1=0;$$

$$\therefore\ \tan\theta=\frac{2n+1\pm\sqrt{(2n+1)^2-16}}{4}$$

$$=\frac{2n+1}{4}\pm\frac{\sqrt{4n^2+4n-15}}{4}.$$

19. $\tan^{-1}x+\cot^{-1}y=\tan^{-1}3;$

$$\therefore\ \frac{x+\frac{1}{y}}{1-\frac{x}{y}}=3;$$

$$\therefore\ y=\frac{3x+1}{3-x};$$

thus we see that the only positive integral values which x may have are $1, 2$;

when $x=1,\ y=2;$

when $x=2,\ y=7;$

and these are all the positive integral solutions of the equation.

MISCELLANEOUS EXAMPLES. G. PAGE 246.

1. Since $$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c},$$

we have $$\frac{a}{4}=\frac{b}{5}=\frac{c}{6};$$

$$\therefore\ \cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{45}{60}=\frac{12}{16}.$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}=\frac{27}{48}=\frac{9}{16}.$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{5}{40}=\frac{2}{16}.$$

That is, the cosines are in the ratio of $12 : 9 : 2$.

2. (1) $$2\cos^3\theta+\sin^2\theta-1=0;$$

$$\therefore\ 2\cos^3\theta=1-\sin^2\theta=\cos^2\theta;$$

$$\therefore\ \cos\theta=0,\ \text{or}\ \frac{1}{2};$$

$$\therefore\ \theta=\frac{(2n+1)\pi}{2},\ \text{or}\ 2n\pi\pm\frac{\pi}{3}.$$

(2) $$\sec^3\theta-2\tan^2\theta=2;$$

$$\therefore\ \sec^3\theta=2(1+\tan^2\theta)=2\sec^2\theta;$$

$$\therefore\ \sec\theta=2,\ \text{since}\ \sec\theta=0\ \text{is inadmissible};$$

$$\therefore\ \theta=2n\pi\pm\frac{\pi}{3}.$$

3. $$\tan\beta=2\sin\alpha\sin\gamma\operatorname{cosec}(\alpha+\gamma);$$

$$\therefore\ 2\cot\beta=\frac{\sin(\alpha+\gamma)}{\sin\alpha\sin\gamma}=\cot\alpha+\cot\gamma;$$

$\therefore\ \cot\alpha, \cot\beta, \cot\gamma$ are in arithmetical progression.

4. $$4r(r_1+r_2+r_3)=\frac{4\Delta^2}{s}\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)$$
$$=4\{(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)\}$$
$$=4\{3s^2-2(a+b+c)s+bc+ca+ab\}$$
$$=4(bc+ca+ab)-4s^2$$
$$=4(bc+ca+ab)-(a+b+c)^2$$
$$=2(bc+ca+ab)-(a^2+b^2+c^2).$$

5. (1) $$\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{1}{3}-\frac{1}{5}}{1+\frac{1}{15}}+\tan^{-1}\frac{1}{7}$$
$$=\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{7}$$
$$=\tan^{-1}\frac{\frac{1}{8}+\frac{1}{7}}{1-\frac{1}{56}}=\tan^{-1}\frac{3}{11}.$$

(2) $$\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}\right)=\frac{4}{5}\cdot\frac{15}{17}-\frac{3}{5}\cdot\frac{8}{17}$$
$$=\frac{36}{85};$$
$$\therefore\ \sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{36}{85}$$
$$=\frac{\pi}{2}-\sin^{-1}\frac{36}{85}.$$

6. $$\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}=\sqrt{\frac{333\times74}{185\times222}}=\sqrt{\frac{6}{10}}.$$
$$\log\cos\frac{C}{2}=\frac{1}{2}(\log6-1)=\frac{1}{2}(\bar{1}\cdot7781513)$$
$$=\bar{1}\cdot8890756;$$
$$\log\cos 39^\circ 14'=\bar{1}\cdot8890644$$
$$\text{diff.}\qquad\qquad 112$$
$$\text{propl. } decrease=\frac{112}{1032}\times60''=6\cdot5''.$$
$$\therefore\ \frac{C}{2}=39^\circ 13'\,53\cdot5'',\ \text{and}\ C=78^\circ 27'\,47''.$$

7.
$$\tan(\alpha+\theta)=n\tan(\alpha-\theta);$$
$$\therefore\ \frac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)}=n;$$
$$\therefore\ \frac{\tan(\alpha+\theta)-\tan(\alpha-\theta)}{\tan(\alpha+\theta)+\tan(\alpha-\theta)}=\frac{n-1}{n+1};$$
that is,
$$\frac{\sin 2\theta}{\sin 2\alpha}=\frac{n-1}{n+1}.$$

8.
$$8R^2=a^2+b^2+c^2;$$
$$\therefore\ 2=\sin^2 A+\sin^2 B+\sin^2 C$$
$$=2-\cos(A+B)\cos(A-B)-\cos^2 C$$
$$=2+2\cos A\cos B\cos C;$$
$$\therefore\ \cos A\cos B\cos C=0;$$
that is, one of the angles of the triangle is a right angle.

9. Area of inscribed polygon $=nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$;

Area of circumscribed polygon $=nr^2\tan\frac{\pi}{n}$;

$\therefore$ we have
$$\cos^2\frac{\pi}{n}=\frac{3}{4};$$
$$\therefore\ \cos\frac{\pi}{n}=\frac{\sqrt{3}}{2};$$
$$\therefore\ n=6.$$

10. Let P, Q be the positions of the boats; then we have
$$\angle PBA=\angle QBC=45°,\ \angle PCB=15°,\ \angle QCD=75°.$$
$$\therefore\ PB=\frac{400\sin 15°}{\sin 30°}=200\sqrt{2}(\sqrt{3}-1);$$
$$QB=\frac{400\sin 75°}{\sin 30°}=200\sqrt{2}(\sqrt{3}+1).$$
But PBQ is a right angle;
$$\therefore\ PQ^2=PB^2+QB^2=200^2(12+4)=200^2\times 4^2;$$
$$\therefore\ PQ=800.$$
Again, the distance of P from $AB=PB\sin 45°=200(\sqrt{3}-1)$
$$=146{\cdot}4\text{ yds.},$$
and the distance of Q from $AB=QB\sin 75°=200(\sqrt{3}+1)$
$$=546{\cdot}4\text{ yds.}$$

EXAMPLES. XX. a. PAGE 255.

1. When $\frac{A}{2}$ lies between $-135°$ and $-180°$, $\sin\frac{A}{2}$ is negative, therefore in the first formula of Art. 254, the negative sign must be taken.

2. $\frac{A}{2}$ lies between 135° and 180°; and therefore

$$\sin\frac{A}{2} \text{ is positive, } \cos\frac{A}{2} \text{ is negative.}$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\frac{119}{169}}{2}} = \frac{5}{13},$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\sqrt{\frac{1+\frac{119}{169}}{2}} = -\frac{12}{13}.$$

3. Here $\sin\frac{A}{2}$ is negative, and $\cos\frac{A}{2}$ is positive;

$$\therefore \sin\frac{A}{2} = -\sqrt{\frac{1-\cos A}{2}} = -\sqrt{\frac{1+\frac{161}{289}}{2}} = -\frac{15}{17},$$

$$\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}} = \sqrt{\frac{1-\frac{161}{289}}{2}} = \frac{8}{17}.$$

4. When $\frac{A}{2}$ lies between 135° and 225°, $\cos\frac{A}{2} > \sin\frac{A}{2}$ and is negative;

$$\therefore \sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A},$$

and

$$\sin\frac{A}{2} - \cos\frac{A}{2} = +\sqrt{1+\sin A}.$$

7. When $\frac{A}{2}$ lies between 45° and 90°, $\sin\frac{A}{2} > \cos\frac{A}{2}$ and is positive;

$$\therefore \sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{1+\sin A} = \sqrt{1+\frac{24}{25}} = \frac{7}{5},$$

and

$$\sin\frac{A}{2} - \cos\frac{A}{2} = \sqrt{1-\sin A} = \sqrt{1-\frac{24}{25}} = \frac{1}{5};$$

$$\therefore \sin\frac{A}{2} = \frac{4}{5}, \cos\frac{A}{2} = \frac{3}{5}.$$

8. When $\frac{A}{2}$ lies between 135° and 180°, $\cos\frac{A}{2} > \sin\frac{A}{2}$ and is negative;

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2} = -\sqrt{1+\sin A} = -\sqrt{1-\frac{240}{289}} = -\frac{7}{17},$$

and
$$\sin\frac{A}{2}-\cos\frac{A}{2} = \sqrt{1-\sin A} = \sqrt{1+\frac{240}{289}} = \frac{23}{17};$$

$$\therefore \sin\frac{A}{2} = \frac{8}{17}, \cos\frac{A}{2} = -\frac{15}{17}.$$

9. (1) We have

$$\sin A+\cos A = \sqrt{1+\sin 2A} \quad \text{.....(i)},$$

$$\sin A-\cos A = -\sqrt{1-\sin 2A} \quad \text{.....(ii)}.$$

From (i) we see that of $\sin A$ and $\cos A$ *the numerically greater is positive.*

From (ii) we see that $\cos A$ is the greater.

Now $\cos A$ is greater than $\sin A$ and positive between the limits $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$.

(3) We have

$$\sin A+\cos A = -\sqrt{1+\sin 2A} \quad \text{.....(i)},$$

$$\sin A-\cos A = +\sqrt{1-\sin 2A} \quad \text{.....(ii)}.$$

From (i) we see that of $\sin A$ and $\cos A$ the numerically greater is negative.

From (ii) we see that $\cos A$ is the greater.

Now $\cos A$ is greater than $\sin A$ and negative between the limits $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$.

10. If $\frac{A}{2}=120°$, $\sin\frac{A}{2} > \cos\frac{A}{2}$ and is positive;

$$\therefore \sin\frac{A}{2}+\cos\frac{A}{2} = \sqrt{1+\sin A},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2} = \sqrt{1-\sin A}.$$

$$\therefore 2\sin\frac{A}{2} = \sqrt{1+\sin A}+\sqrt{1-\sin A}.$$

11. $\tan 7\frac{1}{2}^\circ = \dfrac{1-\cos 15^\circ}{\sin 15^\circ} = \operatorname{cosec} 15^\circ - \cot 15^\circ$

$$= \frac{2\sqrt{2}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} - (2+\sqrt{3})$$

$$= \sqrt{6}+\sqrt{2}-2-\sqrt{3};$$

$$\cot 142\tfrac{1}{2}^\circ = \frac{1+\cos 285^\circ}{\sin 285^\circ} = \operatorname{cosec} 285^\circ + \cot 285^\circ$$

$$= -\operatorname{cosec} 75^\circ - \cot 75^\circ = -\frac{2\sqrt{2}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} - (2-\sqrt{3})$$

$$= -\sqrt{2}(\sqrt{3}-1) - 2 + \sqrt{3} = \sqrt{2}+\sqrt{3}-2-\sqrt{6}.$$

12. $\sin 9^\circ = \frac{1}{4}\{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\}$ [Art. 260.]

$$= \frac{1}{4}\{\sqrt{5{\cdot}236080} - \sqrt{2{\cdot}7639320}\}$$

$$= \frac{1}{4}\{2{\cdot}288\ldots - 1{\cdot}662\ldots\}$$

$$= \frac{{\cdot}626\ldots}{4} = {\cdot}156\ldots.$$

13. (1) As in Art. 251, $2\cos\frac{\pi}{8} = \sqrt{2+\sqrt{2}}$;

but $4\sin^2\frac{\pi}{16} = 2 - 2\cos\frac{\pi}{8} = 2 - \sqrt{2+\sqrt{2}}$;

$$\therefore\ 2\sin\frac{\pi}{16} = \sqrt{2-\sqrt{2+\sqrt{2}}}.$$

(2) $\tan 11^\circ 15' = \dfrac{1-\cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \operatorname{cosec} 22\tfrac{1}{2}^\circ - \cot 22\tfrac{1}{2}^\circ$

$$= \frac{2}{\sqrt{2-\sqrt{2}}} - \frac{1}{\sqrt{2-1}} = \frac{2\sqrt{2+\sqrt{2}}}{\sqrt{2}} - (\sqrt{2}+1)$$

$$= \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1).$$

14. (1) $\cos\theta + \sin\theta = \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$.

As θ increases from 0 to $\frac{\pi}{4}$, the expression is positive and increases from 1 to $\sqrt{2}$.

As θ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, the expression is positive and decreases from $\sqrt{2}$ to 0.

As θ increases from $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$, the expression is negative and increases numerically from 0 to $-\sqrt{2}$.

As θ increases from $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$, the expression is negative and decreases numerically from $-\sqrt{2}$ to 0.

As θ increases from $\frac{7\pi}{4}$ to 2π, the expression is positive and increases from 0 to 1.

(2) $\sin\theta - \sqrt{3}\cos\theta = 2\left(\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right) = 2\sin\left(\theta - \frac{\pi}{3}\right)$.

From $\theta = 0$ to $\frac{\pi}{3}$, the expression is negative and decreases numerically from $-\sqrt{3}$ to 0.

From $\theta = \frac{\pi}{3}$ to $\frac{\pi}{2} + \frac{\pi}{3}$, or $\frac{5\pi}{6}$, the expression is positive and increases from 0 to 2.

From $\theta = \frac{5\pi}{6}$ to $\frac{\pi}{2} + \frac{5\pi}{6}$, or $\frac{4\pi}{3}$, the expression is positive and decreases from 2 to 0.

From $\theta = \frac{4\pi}{3}$ to $\frac{\pi}{2} + \frac{4\pi}{3}$, or $\frac{11\pi}{6}$, the expression is negative and increases numerically from 0 to -2.

From $\theta = \frac{11\pi}{6}$ to 2π, the expression is negative and decreases numerically from -2 to $-\sqrt{3}$.

15. (1) The expression $= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta - \cos^2\theta} = -\frac{1}{\cos 2\theta} = -\sec 2\theta$

$= -\sec\phi$, where $\phi = 2\theta$.

We have therefore to trace the changes in $-\sec\phi$, from $\phi = 0$ to 2π.

From 0 to $\frac{\pi}{2}$, the expression is negative and increases numerically from -1 to $-\infty$.

From $\frac{\pi}{2}$ to π, the expression is positive and decreases from ∞ to 1.

From π to $\frac{3\pi}{2}$, the expression is positive and increases from 1 to ∞.

From $\frac{3\pi}{2}$ to 2π, the expression is negative and decreases numerically from $-\infty$ to -1.

(2) The expression $= \frac{2\sin\theta(1-\cos\theta)}{2\sin\theta(1+\cos\theta)} = \tan^2\frac{\theta}{2}$.

Now $\frac{\theta}{2}$ varies from 0 to $\frac{\pi}{2}$, so that the expression is positive and increases from 0 to ∞.

EXAMPLES. XX. b. PAGE 260.

1. Here $\tan\frac{A}{2}$ is negative; hence in the formula $\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$, the numerator and denominator must have different signs. But when $A=320°$, $\tan A$ is negative; therefore we must take the sign which will make the numerator positive,

$$\therefore \tan\frac{A}{2} = \frac{-1+\sqrt{1+\tan^2 A}}{\tan A}.$$

3. $\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$, and since $\tan A$ is positive the radical must have the positive sign prefixed.

$$\therefore \tan A = \sqrt{\frac{13-12}{13+12}} = \frac{1}{5}.$$

4. $\cot\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{1-\cos A}}$; and when $\frac{A}{2}$ lies between 90° and 135°, $\cot\frac{A}{2}$ is negative;

$$\therefore \cot\frac{A}{2} = -\sqrt{\frac{5-4}{5+4}} = -\frac{1}{3}.$$

5. The general solution of $\cot 2\theta = \cot 2\alpha$ is

$$2\theta = n\pi + 2\alpha; \text{ therefore } \theta = \frac{1}{2}(n\pi + 2\alpha),$$

(1) when $n=2m$, $\quad \cot\theta = \cot(m\pi+\alpha) = \cot\alpha$,

(2) when $n=2m+1$, $\quad \cot\theta = \cot\left(m\pi+\frac{\pi}{2}+\alpha\right) = -\tan\alpha$.

6. $\sin\theta = \sin\alpha$; $\therefore \theta = n\pi + (-1)^n\alpha$; hence in finding $\sin\frac{\theta}{3}$ we have to consider all the values of $\sin\left(\frac{n\pi}{3} + (-1)^n\frac{\alpha}{3}\right)$. Give to n in succession the values 0, 1, 2, 3,.... Proceeding as in Art. 264 we shall find that the first six angles are

$$\frac{\alpha}{3},\ \frac{\pi}{3}-\frac{\alpha}{3},\ \frac{2\pi}{3}+\frac{\alpha}{3},\ \pi-\frac{\alpha}{3},\ \frac{4\pi}{3}+\frac{\alpha}{3},\ \frac{5\pi}{3}-\frac{\alpha}{3},$$

and that the other angles are coterminal with one of these.

Now $\sin\frac{2\pi}{3}+\frac{\alpha}{3}=\sin\left\{\pi-\left(\frac{\pi}{3}-\frac{\alpha}{3}\right)\right\}=\sin\frac{\pi-\alpha}{3}$;

$\sin\left(\pi-\frac{\alpha}{3}\right)=\sin\frac{\alpha}{3}$;

$\sin\left(\frac{4\pi}{3}+\frac{\alpha}{3}\right)=\sin\left\{\pi+\left(\frac{\pi}{3}+\frac{\alpha}{3}\right)\right\}=-\sin\frac{\pi+\alpha}{3}$;

$\sin\left(\frac{5\pi}{3}-\frac{\alpha}{3}\right)=\sin\left\{2\pi-\left(\frac{\pi}{3}+\frac{\alpha}{3}\right)\right\}=-\sin\frac{\pi+\alpha}{3}$.

Thus the values of $\sin\frac{\theta}{3}$ are $\sin\frac{\alpha}{3}$, $\sin\frac{\pi-\alpha}{3}$, $-\sin\frac{\pi+\alpha}{3}$.

7. Here $\theta=n\pi+\alpha$, and we have to find all the values of $\tan\left(\frac{n\pi}{3}+\frac{\alpha}{3}\right)$.

Give to n in succession the values 0, 1, 2, 3,...; then we shall find that all the angles are coterminal with one of the following:

$$\frac{\alpha}{3},\ \frac{\pi}{3}+\frac{\alpha}{3},\ \frac{2\pi}{3}+\frac{\alpha}{3},\ \pi+\frac{\alpha}{3},\ \frac{4\pi}{3}+\frac{\alpha}{3},\ \frac{5\pi}{3}+\frac{\alpha}{3},$$

and as in Ex. 6 it may be shewn that the tangents of these angles assume one of the three forms

$$\tan\frac{\alpha}{3},\ \tan\frac{\pi+\alpha}{3},\ -\tan\frac{\pi-\alpha}{3}.$$

8. Here $3\theta=2n\pi\pm3\alpha$, or $\theta=\frac{2n\pi}{3}\pm\alpha$, and we have to find all the values of $\sin\left(\frac{2n\pi}{3}\pm\alpha\right)$.

Now n must be of the form $3m$, or $3m+1$, or $3m-1$.

If $n=3m$, $\sin\left(\frac{2n\pi}{3}\pm\alpha\right)=\sin(2m\pi\pm\alpha)=\pm\sin\alpha$;

if $n=3m+1$, $\sin\left(\frac{2n\pi}{3}\pm\alpha\right)=\sin\left(2m\pi+\frac{2\pi}{3}\pm\alpha\right)=\sin\left(\frac{2\pi}{3}\pm\alpha\right)$;

if $n=3m-1$, $\sin\left(\frac{2n\pi}{3}\pm\alpha\right)=\sin\left(2m\pi-\frac{\pi}{3}\pm\alpha\right)=-\sin\left(\frac{\pi}{3}\pm\alpha\right)$.

9. Here $3\theta=n\pi+(-1)^n3\alpha$, or $\theta=\frac{n\pi}{3}+(-1)^n\alpha$, and we have to find all the values of $\cos\left\{\frac{n\pi}{3}+(-1)^n\alpha\right\}$.

If $n=3m$, $\cos\left\{\frac{n\pi}{3}+(-1)^n\alpha\right\}=\cos\{m\pi+(-1)^{3m}\alpha\}=\pm\cos\alpha$;

if $n=3m+1$, $\cos\left\{\frac{n\pi}{3}+(-1)^n\alpha\right\}=\cos\left\{m\pi+\frac{\pi}{3}+(-1)^{3m+1}\alpha\right\}$

$$=\pm\cos\left(\frac{\pi}{3}\pm\alpha\right);$$

if $n=3m+2$, $\cos\left\{\frac{n\pi}{3}+(-1)^n\alpha\right\}=\cos\left\{m\pi+\frac{2\pi}{3}+(-1)^{3m+2}\alpha\right\}$

$$=\pm\cos\left(\frac{2\pi}{3}\pm\alpha\right).$$

EXAMPLES. XXI. a. PAGE 267.

1. Let $x=$distance in feet, then $x=44\cot 35'=\frac{44}{\theta}$, nearly, where θ is the radian measure of $35'$.

That is, $$x=44\times\frac{60}{35}\times\frac{180}{\pi}=\frac{44\times 6\times 180}{11}\text{ ft.};$$

whence $x=1440$ yds.

2. Here $$x=\frac{22}{3}\cot 24'\,30''=\frac{22}{3}\times\frac{180\times 60}{77}\text{ ft., nearly}$$

$$=342\tfrac{6}{7}\text{ yds.}$$

3. With the figure of Ex. 1, p. 264, we have

$$PN=840\tan 1°\,30'=840\times\frac{3}{2}\times\frac{\pi}{180}\text{ yds., nearly}$$

$$=840\times\frac{11}{420}=22\text{ yds.}$$

4. Let θ be the required angle, then approximately

$$\theta=\tan\theta=\frac{121}{1760\times 3\times 12}=\frac{11}{160\times 3\times 12}\text{ radians}$$

$$=\frac{11}{160\times 3\times 12}\times\frac{180\times 7}{22}\times 60\text{ minutes}$$

$$=6'\,34''.$$

5. Let θ be the required angle, then approximately

$$\frac{\theta}{2}=\tan\frac{\theta}{2}=\frac{2}{3000}=\frac{1}{1500}\text{ radians};$$

$$\therefore\ \theta=\frac{1}{750}\times\frac{180\times 7}{22}\times 60\text{ minutes}=4'\,35''.$$

6. The radian measure of $\frac{1^\circ}{4} = \frac{22}{4\times180\times7} = \frac{11}{14\times180}$;

$$\therefore\ x = \cdot625 \cot\frac{1^\circ}{4} = \frac{14\times180}{11}\times\cdot625 \text{ inches, nearly}$$

$$= 11\text{ ft. }11\text{ in.}$$

7. The radian measure of $5' = \frac{5\times22}{60\times180\times7} = \frac{11}{42\times180}$;

$$\therefore\ x = \frac{11}{2}\cot\frac{5'}{2} = \frac{11}{2}\times\frac{84\times180}{11} \text{ inches, nearly}$$

$$= 210\text{ yards.}$$

8. Let θ be the difference between the latitudes, then

$$\theta = \frac{11}{3960} = \frac{1}{360} \text{ radians}$$

$$= \frac{180\times7\times60}{360\times22} \text{ minutes} = 9'\,33''.$$

9. See figure of Example 2, page 264.

Let DC be the man, CB the tower, A the point distant 24 feet from the tower;

Let $\angle BAC = \alpha$, $\angle CAD = \theta$;

then
$$\tan\alpha = \frac{120}{24} = 5,$$

$$\tan(\alpha+\theta) = \frac{126}{24} = \frac{21}{4}.$$

But $$\tan(\alpha+\theta) = \frac{\tan\alpha+\tan\theta}{1-\tan\alpha\tan\theta} = \frac{\tan\alpha+\theta}{1-\theta\tan\alpha}, \text{ approximately;}$$

$$\therefore \frac{21}{4} = \frac{5+\theta}{1-5\theta};$$

whence
$$\theta = \frac{1}{109} \text{ radians} = 31\cdot5', \text{ nearly.}$$

10. See figure of Example 2, page 264.

Let DC be the flagstaff, CB the cliff, and let $DC = x$ feet; then taking the angles as before, we have

$$\tan\alpha = \frac{490}{980} = \frac{1}{2} = \cdot5,\ \tan\theta = \cdot04 \text{ approximately;}$$

$$\therefore \tan(\alpha+\theta) = \frac{\cdot5+\cdot04}{1-\cdot02} = \frac{54}{98} = \frac{27}{49};$$

$$\therefore \frac{x+490}{980} = \frac{27}{49};$$

whence
$$x = 50;$$

that is, the height of the flagstaff is 50 feet.

11. (1) $n' = \dfrac{n\pi}{180 \times 60}$ radians;

$$\therefore \underset{n=0}{Lt.}\left(\frac{\sin n'}{n}\right) = \frac{n\pi}{180 \times 60} \times \frac{1}{n} = \frac{\pi}{10800}.$$

(2) $n'' = \dfrac{n\pi}{180 \times 60 \times 60}$ radians;

$$\therefore \underset{n=0}{Lt.}\left(\frac{\sin n''}{n}\right) = \frac{n\pi}{180 \times 60 \times 60} \times \frac{1}{n} = \frac{\pi}{648000}.$$

12. $\dfrac{1}{2} nr^2 \sin\dfrac{2\pi}{n} = \pi r^2 \cdot \dfrac{n}{2\pi} \cdot \sin\dfrac{2\pi}{n} = \pi r^2 \left(\sin\dfrac{2\pi}{n} \div \dfrac{2\pi}{n}\right)$;

but when $n = \infty$, $\dfrac{2\pi}{n} = 0$, therefore the limit of $\sin\dfrac{2\pi}{n} \div \dfrac{2\pi}{n}$ is unity.

Thus the required limit is πr^2.

13. $\underset{\theta=0}{Lt.}\left(\dfrac{1-\cos\theta}{\theta \sin\theta}\right) = \underset{\theta=0}{Lt.}\left(\dfrac{\tan\frac{\theta}{2}}{\theta}\right) = \underset{\theta=0}{Lt.}\left(\dfrac{1}{2} \cdot \dfrac{\tan\frac{\theta}{2}}{\frac{\theta}{2}}\right) = \dfrac{1}{2}$.

14. $\underset{\theta=0}{Lt.}\left(\dfrac{m \sin m\theta - n \sin n\theta}{\tan m\theta + \tan n\theta}\right) = \underset{\theta=0}{Lt.}\left(\dfrac{m \cdot m\theta - n \cdot n\theta}{m\theta + n\theta}\right)$ [Art. 268.]

$$= \frac{m^2 - n^2}{m + n} = m - n.$$

15. $\cos\left(\dfrac{\pi}{3} + \theta\right) = \dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta = \dfrac{1}{2} - \dfrac{\sqrt{3}}{200} = \cdot 491$ nearly.

16. $10'\,30'' = \dfrac{21}{2 \times 60 \times 180} \times \dfrac{22}{7} = \dfrac{11}{3600}$ radians;

$$\therefore \sin 30°\,10'\,30'' = \sin\left(\frac{\pi}{6} + \frac{11}{3600}\right)$$

$$= \frac{1}{2}\cos\frac{11}{3600} + \frac{\sqrt{3}}{2}\sin\frac{11}{3600}$$

$$= \frac{1}{2} + \frac{11\sqrt{3}}{7200} = \cdot 503 \text{ nearly.}$$

17. $\cos\left(\frac{\pi}{3}+\theta\right)=\cdot49$; and $\cos\frac{\pi}{3}=\cdot5$.

$\therefore$ θ is a very small angle, so that approximately

$$\cos\left(\frac{\pi}{3}+\theta\right)=\frac{1}{2}-\frac{\sqrt{3}}{2}\theta;$$

$$\therefore \frac{1}{2}-\frac{\sqrt{3}}{2}\theta=\cdot49;$$

$$\therefore \frac{\sqrt{3}\theta}{2}=\cdot01;$$

$$\therefore \theta=\frac{1}{50\sqrt{3}}=\frac{\sqrt{3}}{150} \text{ radians}$$

$$=\frac{\sqrt{3}}{150}\times\frac{7}{22}\times180^\circ=\frac{21\sqrt{3}}{55} \text{ degrees}$$

$$=39\cdot7' \text{ nearly.}$$

EXAMPLES. XXI. b. PAGE 271.

1. Let the distance be x miles; then by the rule on page 269, we have

$$x^2=\frac{3\times96}{2}=3\times48;$$

$$\therefore x=12;$$

that is, the distance is 12 miles.

2. Let a feet be the height of lighthouse above the sea level;

then $$15^2=\frac{3a}{2}, \text{ or } a=150;$$

that is, the height of lighthouse = 150 feet.

3. Let the distances in miles of the horizon visible from the masts of the ships be x_1, x_2;

then $$x_1^2=\frac{3\times32\frac{2}{3}}{2}=49; \quad \therefore x_1=7,$$

$$x_2^2=\frac{3\times42\frac{2}{3}}{2}=64; \quad \therefore x_2=8;$$

$\therefore$ the greatest distance at which one mast can be seen from the other

$$=x_1+x_2=15 \text{ miles.}$$

4. Let the distances in miles of the horizon seen from the two masts be x, y respectively,

then $$x^2=\frac{3\times 54}{2}=81;\ \therefore\ x=9;$$

also $$x+y=20;\ \text{whence } y=11.$$

Height of mast of second ship $=\frac{2y^2}{3}$ feet

$$=\frac{242}{3}\text{ feet}=80\text{ ft. }8\text{ in.}$$

5. Let x, y be the distances in miles of the horizon visible from the mast and the lamp respectively;

then $$x^2=\frac{3\times 73\frac{1}{2}}{2};\ \therefore\ x=\frac{21}{2},$$

and $$x+y=28;\ \text{whence } y=\frac{35}{2}.$$

$\therefore$ height of lamp $$=\frac{2y^2}{3}=\frac{35^2}{6}=\frac{1225}{6}\text{ ft.}$$

$$=204\text{ ft. }2\text{ in.}$$

6. From the formula on page 271, we have

$$\text{number of degrees in dip of horizon}=\frac{10}{11}\sqrt{\frac{2\times 2640}{3\times 1760}}=\frac{10}{11};$$

$\therefore$ dip of the horizon $=54'\,33''$, nearly.

7. The greatest distance at which the light must be visible is the distance of a point exactly opposite a point on the shore midway between two lighthouses, and $3\frac{1}{2}$ miles from it.

$$\text{This distance}=\sqrt{12^2+\left(\frac{7}{2}\right)^2}=\sqrt{\frac{625}{4}}=\frac{25}{2}\text{ miles.}$$

$$\therefore\ \text{height of lamp}=\frac{2}{3}\times\frac{625}{4}=\frac{625}{6}\text{ feet}=104\text{ ft. }2\text{ in.}$$

8. Let the height of the hill be h miles,

then we have $$1\cdot8\dot{1}=\frac{10}{11}\sqrt{2h};$$

$$\therefore\ \frac{20}{11}=\frac{10}{11}\sqrt{2h};\ \text{whence } h=2;$$

that is, the height of the hill $=2$ miles $=10560$ feet.

9. Height of hill $= \frac{2 \times (30{\cdot}25)^2}{3}$ feet $= 610$ ft. nearly.

The dip of the horizon $= \frac{10}{11}\sqrt{\frac{4 \times 121^2}{3 \times 16} \times \frac{1}{3 \times 1760}}$ degrees

$$= \frac{5}{12}\sqrt{\frac{11}{10}} \text{ degrees}$$

$$= \frac{5}{2}\sqrt{110} \text{ minutes}$$

$= 26' 13''$, nearly.

10. We have $N = \frac{10}{11}\sqrt{2h}$, where h = height in miles,

$$= \frac{10}{11}\sqrt{\frac{2a}{3 \times 1760}}, \text{ where } a = \text{height in feet},$$

$$= \frac{10}{11}\sqrt{\frac{4x^2}{9 \times 1760}}$$

$$= \frac{x}{66}\sqrt{\frac{10}{11}}.$$

11. $$\frac{\sin 4\theta \cot \theta}{\text{vers } 2\theta \cot^2 2\theta} = \frac{2 \sin 2\theta \cos 2\theta \cot \theta}{2 \sin^2 \theta \cot^2 2\theta}$$

$$= \frac{\sin^3 2\theta \cos \theta}{\sin^3 \theta \cos 2\theta} = \frac{8 \cos^4 \theta}{\cos 2\theta};$$

$$\therefore \underset{\theta=0}{Lt.}\left(\frac{\sin 4\theta \cot \theta}{\text{vers } 2\theta \cot^2 2\theta}\right) = \underset{\theta=0}{Lt.}\left(\frac{8 \cos^4 \theta}{\cos 2\theta}\right) = 8.$$

12. $$\frac{1 - \cos\theta + \sin\theta}{1 - \cos\theta - \sin\theta} = \frac{2 \sin^2 \frac{\theta}{2} + \sin\theta}{2 \sin^2 \frac{\theta}{2} - \sin\theta} = \frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}};$$

$$\therefore \underset{\theta=0}{Lt.}\left(\frac{1 - \cos\theta + \sin\theta}{1 - \cos\theta - \sin\theta}\right) = \underset{\theta=0}{Lt.}\left(\frac{\sin \frac{\theta}{2} + \cos \frac{\theta}{2}}{\sin \frac{\theta}{2} - \cos \frac{\theta}{2}}\right) = -1.$$

13. (1) $\underset{\theta=\alpha}{Lt.}\left(\frac{\sin\theta-\sin\alpha}{\theta-\alpha}\right)=\underset{\theta=\alpha}{Lt.}\left(\frac{\sin\frac{\theta-\alpha}{2}\cos\frac{\theta+\alpha}{2}}{\frac{\theta-\alpha}{2}}\right)$.

But $$\underset{\theta=\alpha}{Lt.}\left(\frac{\sin\frac{\theta-\alpha}{2}}{\frac{\theta-\alpha}{2}}\right)=1,$$ [Art. 266.]

$$\therefore \text{ required limit}=\underset{\theta=\alpha}{Lt.}\left(\cos\frac{\theta+\alpha}{2}\right)=\cos\alpha.$$

(2) $$\underset{\theta=\alpha}{Lt.}\left(\frac{\cos\theta-\cos\alpha}{\theta-\alpha}\right)=\underset{\theta=\alpha}{Lt.}\left(\frac{\sin\frac{\alpha-\theta}{2}\sin\frac{\alpha+\theta}{2}}{\frac{\theta-\alpha}{2}}\right)$$

$$=\underset{\theta=\alpha}{Lt.}\left(-\sin\frac{\alpha+\theta}{2}\right)=-\sin\alpha.$$

14. Let $AB=32$, $AC=31$;

then $$\tan C=\frac{32}{31}=1+\frac{1}{31};$$

$\therefore$ C is a little greater than 45°;

$$\therefore C=\frac{\pi}{4}+\theta, \text{ where } \theta \text{ is small};$$

$$\therefore \frac{1+\theta}{1-\theta}=\frac{32}{31};$$

$$\therefore \theta=\frac{1}{63}\text{ radians}=\frac{1}{63}\times\frac{7\times180}{22}\text{ degrees}=\frac{10^\circ}{11}=54'\,33'';$$

$$\therefore C=45^\circ\,54'\,33'',\ B=44^\circ\,5'\,27''.$$

15. We have $$\angle BPA=\alpha=\angle BAP;$$

$$\therefore AB=BP;$$

$$\therefore \frac{AB}{BC}=\frac{BP}{BC}=\frac{\sin3\alpha}{\sin\alpha}=3-4\sin^2\alpha=3,$$

since the object is distant and therefore α is small;

that is, $$AB=3BC, \text{ nearly.}$$

16. We shall first shew that $\frac{\tan(\theta+h)}{\theta+h} - \frac{\tan\theta}{\theta}$ is positive, h being the radian measure of a small positive angle.

$$\text{This fraction} = \frac{\theta\tan(\theta+h) - \overline{\theta+h}\tan\theta}{\theta(\theta+h)} = \frac{\theta(\tan\overline{\theta+h} - \tan\theta) - h\tan\theta}{\theta(\theta+h)}$$

$$= \frac{\dfrac{\theta\sin h}{\cos\theta\cos(\theta+h)} - \dfrac{h\sin\theta}{\cos\theta}}{\theta(\theta+h)} = \frac{\theta\sin h - h\sin\theta\cos(\theta+h)}{\theta(\theta+h)\cos\theta\cos(\theta+h)}.$$

Now $\frac{\sin h}{h} > \frac{\sin\theta}{\theta}$ if $h<\theta$ [Art. 272.]

and $\cos(\theta+h)<1;$

$\therefore\ \theta\sin h > h\sin\theta\cos(\theta+h);$

that is, the fraction is positive, and $\therefore\ \frac{\tan(\theta+h)}{\theta+h} > \frac{\tan\theta}{\theta}.$

$\therefore\ \frac{\tan\theta}{\theta}$ continually increases as θ increases.

When $\theta=0$, $\frac{\tan\theta}{\theta}=1$; when $\theta=\frac{\pi}{2}$, $\frac{\tan\theta}{\theta}=\infty$.

Thus the proposition is established.

MISCELLANEOUS EXAMPLES. H. PAGE 283.

1. $\cos 2\alpha+\cos 2\beta+2\cos(\alpha+\beta) = 2\cos(\alpha+\beta)\{\cos(\alpha-\beta)+1\}$

$$= 4\cos(\alpha+\beta)\cos^2\frac{\alpha-\beta}{2},$$

$$\sin 2\alpha+\sin 2\beta+2\sin(\alpha+\beta) = 2\sin(\alpha+\beta)\{\cos(\alpha-\beta)+1\}$$

$$= 4\sin(\alpha+\beta)\cos^2\frac{\alpha-\beta}{2};$$

$$\therefore\ \text{hypotenuse} = 4\cos^2\frac{\alpha-\beta}{2}\sqrt{\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)} = 4\cos^2\frac{\alpha-\beta}{2}.$$

2. If the in-centre and circum-centre are at equal distances from BC, we have

$$R\cos A = r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2};$$

$$\therefore\ \cos A = \cos A+\cos B+\cos C-1;$$

$$\therefore\ \cos B+\cos C = 1.$$

3. Let θ be the required angle; then we have

$$\tan(45°+\theta)=2;$$

$$\therefore \log\tan(45°+\theta)=\cdot3010300$$

$$\log\tan 63°\,26' \quad =\cdot3009994$$

$$\text{diff.} \qquad\qquad\quad 306$$

$$\therefore \text{propl. increase}=\frac{306}{3159}\times 60''=5\cdot8''=6'', \text{ nearly};$$

$$\therefore 45°+\theta=63°\,26'\,6'';$$

$$\therefore \text{required angle } 18°\,26'\,6''.$$

4. Denote each expression by E; then

$$E^2=(1-\sin^2\alpha)(1-\sin^2\beta)(1-\sin^2\gamma)=\cos^2\alpha\cos^2\beta\cos^2\gamma;$$

that is,
$$E=\pm\cos\alpha\cos\beta\cos\gamma.$$

5. Let p_1, p_2 be the distances of the chords from the centre, and let r be the radius; then

$$p_1=r\cos 36°;$$

similarly
$$p_2=r\cos 72°;$$

$\therefore$ distance between the chords $=p_1-p_2=r(\cos 36°-\cos 72°)$

$$=r\left(\frac{\sqrt5+1}{4}-\frac{\sqrt5-1}{4}\right)=\frac{r}{2}.$$

Also the sum of the squares of the chords $=4r^2\sin^2 36°+4r^2\sin^2 72°$

$$=4r^2\left(1-\frac{3+\sqrt5}{8}+1-\frac{3-\sqrt5}{8}\right).$$

$$=5r^2.$$

6. Let A be the point at which the railways meet, then we have to solve a triangle in which $A=60°$, $a=43$, $b=48$. It is easy to see that the solution is ambiguous; hence from the third figure of Art. 148 we have

$$CD=48\sin 60°=24\sqrt3, \quad AD=48\cos 60°=24.$$

Also
$$B_2D=\sqrt{43^2-(24\sqrt3)^2}=\sqrt{121}=11.$$

$$\therefore AB_1=24+11=35 \text{ miles},$$

$$AB_2=24-11=13 \text{ miles}.$$

7. $$\alpha = \cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b};$$

$$\therefore \cos\alpha = \frac{xy}{ab} - \sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)};$$

$$\therefore \cos^2\alpha - \frac{2xy}{ab}\cos\alpha + \frac{x^2y^2}{a^2b^2} = \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right);$$

$$\therefore 1-\cos^2\alpha = \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2};$$

that is, $$\sin^2\alpha = \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2}.$$

8. We have $p = 4R\cos\frac{A}{2}$, $q = 4R\cos\frac{B}{2}$, $r = 4R\cos\frac{C}{2}$;

$$\therefore \frac{a}{p} = \frac{2R\sin A}{4R\cos\frac{A}{2}} = \sin\frac{A}{2};$$

$$\therefore \frac{a^2}{p^2} + \frac{b^2}{q^2} + \frac{c^2}{r^2} + \frac{2abc}{pqr} = \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 1.$$ [Ex. 13, p. 120.]

9. Let P be the top of the tower, and let x be its height;

then $$PA = \frac{x}{\sin\alpha}, \quad PB = \frac{x}{\sin\beta};$$

and $$PA^2 + OA^2 = PO^2 = PB^2 + OB^2;$$

$$\therefore \frac{x^2}{\sin^2\alpha} + a^2 = \frac{x^2}{\sin^2\beta} + b^2;$$

$$\therefore x^2 = \frac{(a^2-b^2)\sin^2\alpha\sin^2\beta}{\sin^2\alpha - \sin^2\beta} = \frac{(a^2-b^2)\sin^2\alpha\sin^2\beta}{\sin(\alpha+\beta)\sin(\alpha-\beta)};$$

hence the height of the tower $$= \frac{\sqrt{a^2-b^2}\sin\alpha\sin\beta}{\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}}.$$

10. This example follows readily from the results proved in Examples 18, 24, 25 of XVIII. a.

$$r^2 + r_1^2 + r_2^2 + r_3^2 = (r_1+r_2+r_3-r)^2 + 2r(r_1+r_2+r_3) - 2(r_2r_3 + r_3r_1 + r_1r_2)$$

$$= 16R^2 + 2\{(r_1r_2 + rr_3) + (r_2r_3 + rr_1) + (r_3r_1 + rr_2)\} - 4(r_1r_2 + r_2r_3 + r_3r_1)$$

$$= 16R^2 + 2(ab + bc + ca) - 4s^2$$

$$= 16R^2 - a^2 - b^2 - c^2.$$

11. (1) Let $\angle BAD=\theta$; then $\angle CAD=A-\theta$;

$$\therefore \frac{\sin(A-\theta)}{\sin(A+B)}=\frac{CD}{AD}=\frac{BD}{AD}=\frac{\sin\theta}{\sin B};$$

$$\therefore \frac{\sin(A-\theta)}{\sin\theta}=\frac{\sin(A+B)}{\sin B};$$

$$\therefore \cot\theta-\cot A=\cot A+\cot B;$$

that is,
$$\cot BAD=2\cot A+\cot B.$$

(2) Draw AM perpendicular to BC, then

$$2\cot ADC=\frac{2DM}{AM}=\frac{2DC-2MC}{AM}$$

$$=\frac{(BC-MC)-MC}{AM}=\frac{BM}{AM}-\frac{MC}{AM}$$

$$=\cot B-\cot C.$$

12. See fig. of Art. 223.

Then
$$\frac{a}{p}=\frac{BG+GC}{OG}=\frac{BG}{OG}+\frac{GC}{OG}=\tan C+\tan B;$$

$$\therefore \frac{a}{p}+\frac{b}{q}-\frac{c}{r}=2\tan C,$$

and
$$\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=2(\tan A+\tan B+\tan C);$$

$$\therefore 4\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)=8\tan A\tan B\tan C$$

$$=\left(\frac{a}{p}+\frac{b}{q}-\frac{c}{r}\right)\left(\frac{b}{q}+\frac{c}{r}-\frac{a}{p}\right)\left(\frac{c}{r}+\frac{a}{p}-\frac{b}{q}\right).$$

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1. Here the common difference is 2α;

$$\therefore S=\frac{\sin n\alpha}{\sin\alpha}\sin\frac{a+\overline{2n-1}\alpha}{2}=\frac{\sin^2 n\alpha}{\sin\alpha}.$$

2. Here the common difference is $-\beta$;

$$\therefore S=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\frac{a+\overline{a-n-1}\beta}{2}=\frac{\sin\frac{n\beta}{2}\cos\left(a-\frac{n-1}{2}\beta\right)}{\sin\frac{\beta}{2}}.$$

3. Here the common difference is $-\frac{\pi}{n}$;

$$\therefore S=\frac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{2n}}\sin\frac{a+a-(n-1)\frac{\pi}{n}}{2}=\frac{-\cos\left(a+\frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}.$$

4. Here the common difference is $\frac{\pi}{k}$;

$$\therefore S=\frac{\sin\frac{n\pi}{2k}}{\sin\frac{\pi}{2k}}\cos\frac{\frac{\pi}{k}+\frac{n\pi}{k}}{2}=\frac{\sin\frac{n\pi}{2k}\cos\frac{(n+1)\pi}{2k}}{\sin\frac{\pi}{2k}}.$$

5. The common difference is $\frac{2\pi}{19}$ and the number of terms is 9;

$$\therefore S=\frac{\sin\frac{9\pi}{19}}{\sin\frac{\pi}{19}}\cos\frac{9\pi}{19}=\frac{\sin\frac{18\pi}{19}}{2\sin\frac{\pi}{19}}=\frac{\sin\left(\pi-\frac{\pi}{19}\right)}{2\sin\frac{\pi}{19}}=\frac{1}{2}.$$

6. Here the common difference is $\frac{2\pi}{21}$ and the number of terms is 10;

$$\therefore S=\frac{\sin\frac{10\pi}{21}}{\sin\frac{\pi}{21}}\cos\frac{11\pi}{21}=\frac{\sin\frac{22\pi}{21}}{2\sin\frac{\pi}{21}}=-\frac{1}{2}.$$

7. The common difference is $\frac{\pi}{n}$;

$$\therefore S=\frac{\sin\frac{(n-1)\pi}{2n}}{\sin\frac{\pi}{2n}}\sin\frac{\pi+(n-1)\pi}{2n}$$

$$=\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}\sin\frac{\pi}{2}=\cot\frac{\pi}{2n}.$$

8. The common difference is $\frac{2\pi}{n}$;

$$\therefore S = \frac{\sin\frac{(2n-1)\pi}{n}}{\sin\frac{\pi}{n}} \cos\frac{\pi + \{2(2n-1)-1\}\pi}{2n}$$

$$= \frac{\sin\left(2\pi - \frac{\pi}{n}\right)}{\sin\frac{\pi}{n}} \cos\frac{(4n-2)\pi}{2n} = -\cos\frac{\pi}{n}.$$

9. The common difference is $-\alpha$;

$$\therefore S = \frac{\sin n\alpha}{\sin\frac{\alpha}{2}} \sin\frac{n\alpha + (n-\overline{2n-1})\alpha}{2} = \sin n\alpha.$$

10. The series may be written

$$\sin\theta + \sin(\pi + 2\theta) + \sin(2\pi + 3\theta) + \sin(3\pi + 4\theta) + \ldots$$

$$\therefore S = \frac{\sin\frac{n(\pi+\theta)}{2}}{\sin\frac{\pi+\theta}{2}} \sin\frac{\theta + \overline{n-1}\pi + n\theta}{2}$$

$$= \frac{\sin\frac{n(\pi+\theta)}{2} \sin\left\{\frac{(n+1)\theta}{2} + \frac{(n-1)\pi}{2}\right\}}{\sin\frac{\pi+\theta}{2}}.$$

11. The series may be written

$$\cos\alpha + \cos(\pi + \alpha - \beta) + \cos(2\pi + \alpha - 2\beta) + \ldots$$

$$\therefore S = \frac{\sin\frac{n(\pi-\beta)}{2}}{\sin\frac{\pi-\beta}{2}} \cos\frac{\alpha + (n-1)(\pi-\beta) + \alpha}{2}$$

$$= \frac{\sin\frac{n(\pi-\beta)}{2} \cos\left\{\alpha + \frac{(n-1)(\pi-\beta)}{2}\right\}}{\sin\frac{\pi-\beta}{2}}.$$

12. The series may be written

$$\cos\alpha+\cos\left(\frac{\pi}{2}+\alpha-\beta\right)+\cos(\pi+\alpha-2\beta)+\cos\left(\frac{3\pi}{2}+\alpha-3\beta\right)+\ldots$$

$$\therefore S=\frac{\sin\dfrac{n(\pi-2\beta)}{4}}{\sin\dfrac{\pi-2\beta}{4}}\cos\frac{2\alpha+(n-1)\left(\dfrac{\pi}{2}-\beta\right)}{2}$$

$$=\frac{\sin\dfrac{n(\pi-2\beta)}{4}\cos\left\{\alpha+\dfrac{(n-1)(\pi-2\beta)}{4}\right\}}{\sin\dfrac{\pi-2\beta}{4}}.$$

13. $S=\frac{1}{2}\{(\cos\theta-\cos3\theta)+(\cos\theta-\cos5\theta)+(\cos\theta-\cos7\theta)+\ldots\}$

$$=\frac{1}{2}\{n\cos\theta-(\cos3\theta+\cos5\theta+\cos7\theta+\ldots+\cos\overline{2n+1}\,\theta)\}$$

$$=\frac{n\cos\theta}{2}-\frac{\sin n\theta}{2\sin\theta}\cos(n+2)\,\theta.$$

14. $S=\frac{1}{2}\{(\sin4\alpha-\sin2\alpha)+(\sin8\alpha-\sin2\alpha)+(\sin12\alpha-\sin2\alpha)+\ldots\}$

$$=\frac{1}{2}\{(\sin4\alpha+\sin8\alpha+\sin12\alpha+\ldots+\sin4n\alpha)-n\sin2\alpha\}$$

$$=\frac{\sin2n\alpha}{2\sin2\alpha}\sin2(n+1)\alpha-\frac{n\sin2\alpha}{2}.$$

15. $\sec\alpha\sec2\alpha=\dfrac{1}{\cos\alpha\cos2\alpha}=\operatorname{cosec}\alpha\cdot\dfrac{\sin(2\alpha-\alpha)}{\cos\alpha\cos2\alpha}$

$$=\operatorname{cosec}\alpha(\tan2\alpha-\tan\alpha).$$

Similarly, $\sec2\alpha\sec3\alpha=\operatorname{cosec}\alpha(\tan3\alpha-\tan2\alpha)$,

.....

$$\sec n\alpha\sec(n+1)\alpha=\operatorname{cosec}\alpha\{\tan(n+1)\alpha-\tan n\alpha\}.$$

By addition, $S=\operatorname{cosec}\alpha\{\tan(n+1)\alpha-\tan n\alpha\}$.

16. $$\operatorname{cosec}\theta\operatorname{cosec}3\theta=\operatorname{cosec}2\theta\,(\cot\theta-\cot3\theta),$$

$$\operatorname{cosec}3\theta\operatorname{cosec}5\theta=\operatorname{cosec}2\theta\,(\cot3\theta-\cot5\theta),$$

.....

$$\operatorname{cosec}(2n-1)\theta\operatorname{cosec}(2n+1)\theta=\operatorname{cosec}2\theta\,\{\cot(2n-1)\theta-\cot(2n+1)\theta\}.$$

By addition, $$S=\operatorname{cosec}2\theta\,\{\cot\theta-\cot(2n+1)\theta\}.$$

17. $$\tan\frac{a}{2}\sec a=\tan a-\tan\frac{a}{2},$$

$$\tan\frac{a}{2^2}\sec\frac{a}{2}=\tan\frac{a}{2}-\tan\frac{a}{2^2},$$

.....

$$\tan\frac{a}{2^n}\sec\frac{a}{2^{n-1}}=\tan\frac{a}{2^{n-1}}-\tan\frac{a}{2^n}.$$

By addition, $$S=\tan a-\tan\frac{a}{2^n}.$$

18. $$\cos2a\operatorname{cosec}3a=\frac{\cos2a\sin a}{\sin3a\sin a}=\frac{1}{2}\cdot\frac{\sin3a-\sin a}{\sin3a\sin a}$$

$$=\frac{1}{2}(\operatorname{cosec}a-\operatorname{cosec}3a),$$

$$\cos6a\operatorname{cosec}9a=\frac{1}{2}(\operatorname{cosec}3a-\operatorname{cosec}9a),$$

.....

$$\cos3^{n-1}.2a\operatorname{cosec}3^na=\frac{1}{2}(\operatorname{cosec}3^{n-1}a-\operatorname{cosec}3^na).$$

By addition, $$S=\frac{1}{2}\operatorname{cosec}a-\operatorname{cosec}3^na.$$

19. $$\sin a\sec3a=\frac{\sin a}{\cos3a}=\frac{2\sin a\cos a}{2\cos3a\cos a}$$

$$=\frac{\sin(3a-a)}{2\cos3a\cos a}$$

$$=\frac{1}{2}(\tan3a-\tan a),$$

$$\sin3a\sec9a=\frac{1}{2}(\tan9a-\tan3a),$$

.....

$$\sin3^{n-1}a\sec3^na=\frac{1}{2}(\tan3^na-\tan3^{n-1}a).$$

By addition, $$S=\frac{1}{2}(\tan3^na-\tan a).$$

20. Let AB be the diameter, P_1, P_2, $P_3 \ldots P_{n-1}$ the points of division of the arc of the semicircle starting from the end B; then

$$AP_1 = 2a\cos\frac{\pi}{2n},\quad AP_2 = 2a\cos\frac{2\pi}{2n}, \ldots AP_{n-1} = 2a\cos\frac{(n-1)\pi}{2n};$$

$$\therefore \text{ sum of distances} = 2a\left\{\cos\frac{\pi}{2n} + \cos\frac{2\pi}{2n} + \cos\frac{3\pi}{2n} + \ldots + \cos\frac{(n-1)\pi}{2n}\right\}$$

$$= \frac{2a\sin\dfrac{(n-1)\pi}{4n}}{\sin\dfrac{\pi}{4n}}\cos\frac{\pi}{4}$$

$$= \frac{a\left\{\sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{4n}\right) - \sin\dfrac{\pi}{4n}\right\}}{\sin\dfrac{\pi}{4n}}$$

$$= a\left(\cot\frac{\pi}{4n} - 1\right).$$

21. Let P_1, P_2, $P_3 \ldots$ be the angular points of the polygon beginning with that nearest to XX' in the quadrant XOY.

Let p_1, p_2, $p_3 \ldots$ be perpendiculars from P_1, P_2, $P_3 \ldots$ on XX',

and q_1, q_2, $q_3 \ldots$ be perpendiculars from P_1, P_2, $P_3 \ldots$ on YY'.

Let $\angle P_1OX = \theta$, and let r be the radius of the circle;

then $p_1 = r\sin\theta$, $p_2 = r\sin\left(\theta + \dfrac{2\pi}{n}\right)$, $p_3 = r\sin\left(\theta + \dfrac{4\pi}{n}\right)$, $\ldots$;

$$\therefore S_p = r\left\{\sin\theta + \sin\left(\theta + \frac{2\pi}{n}\right) + \sin\left(\theta + \frac{4\pi}{n}\right) + \ldots \text{ to } n \text{ terms}\right\}$$

$$= r\frac{\sin\pi}{\sin\dfrac{\pi}{n}} \cdot \sin\frac{2\theta + (n-1)\dfrac{2\pi}{n}}{2} = 0.$$

Similarly $S_q = 0$.

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1.

$$2S = 1 + \cos 2\theta + 1 + \cos 6\theta + 1 + \cos 10\theta + \ldots$$

$$= n + \frac{\sin 2n\theta}{\sin 2\theta}\cos\frac{2\theta + 2\theta + (n-1)4\theta}{2};$$

$$\therefore S = \frac{n}{2} + \frac{\sin 4n\theta}{4\sin 2\theta}.$$

2. $2S = 1 - \cos 2\alpha + 1 - \cos 2\left(\alpha + \frac{\pi}{n}\right) + 1 - \cos 2\left(\alpha + \frac{2\pi}{n}\right) + \dots$

$$= n - \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{4\alpha + (n-1)\frac{2\pi}{n}}{2} = n.$$

3. $2S = 1 + \cos 2\alpha + 1 + \cos 2\left(\alpha - \frac{\pi}{n}\right) + 1 + \cos 2\left(\alpha - \frac{2\pi}{n}\right) + \dots$

$$= n + \frac{\sin \pi}{\sin \frac{\pi}{n}} \cos \frac{4\alpha - (n-1)\frac{2\pi}{n}}{2} = n.$$

4. $4S = 3 \sin \theta - \sin 3\theta + 3 \sin 2\theta - \sin 6\theta + 3 \sin 3\theta - \sin 9\theta + \dots$

$$= 3\{\sin \theta + \sin 2\theta + \sin 3\theta + \dots\} - (\sin 3\theta + \sin 6\theta + \sin 9\theta + \dots)$$

$$= \frac{3 \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} \sin \frac{(n+1)\theta}{2} - \frac{\sin \frac{3n\theta}{2}}{\sin \frac{3\theta}{2}} \sin \frac{3(n+1)\theta}{2};$$

$$\therefore S = \frac{3 \sin \frac{n\theta}{2} \sin \frac{(n+1)\theta}{2}}{4 \sin \frac{\theta}{2}} - \frac{\sin \frac{3n\theta}{2} \sin \frac{3(n+1)\theta}{2}}{\sin \frac{3\theta}{2}}.$$

5. $4S = 3\left\{\sin \alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \sin\left(\alpha + \frac{4\pi}{n}\right) + \dots\right\}$

$$- \left\{\sin 3\alpha + \sin 3\left(\alpha + \frac{2\pi}{n}\right) + \sin 3\left(\alpha + \frac{4\pi}{n}\right) + \dots\right\}$$

$$= 3 \frac{\sin \pi}{\sin \frac{\pi}{n}} \sin \frac{2\alpha + (n-1)\frac{2\pi}{n}}{2} - \frac{\sin 3\pi}{\sin \frac{3\pi}{n}} \sin \frac{6\alpha + (n-1)\frac{6\pi}{n}}{2}$$

$$= 0.$$

6. $$4S = 3\left\{\cos\alpha + \cos\left(\alpha - \frac{2\pi}{n}\right) + \cos\left(\alpha - \frac{4\pi}{n}\right) + \ldots\right\}$$
$$+ \cos 3\alpha + \cos 3\left(\alpha - \frac{2\pi}{n}\right) + \cos 3\left(\alpha - \frac{4\pi}{n}\right) + \ldots$$
$$= 3\frac{\sin\pi}{\sin\frac{\pi}{n}}\cos\frac{2\alpha - (n-1)\frac{2\pi}{n}}{2} + \frac{\sin 3\pi}{\sin\frac{3\pi}{n}}\cos\frac{6\alpha - (n-1)\frac{6\pi}{n}}{2}$$
$$= 0.$$

7. We have $\tan\theta = \cot\theta - 2\cot 2\theta$, [XI. d. Ex. 18.]

$$2\tan 2\theta = 2\cot 2\theta - 2^2\cot 2^2\theta,$$
$$2^2\tan 2^2\theta = 2^2\cot 2^2\theta - 2^3\cot 2^3\theta,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$2^{n-1}\tan 2^{n-1}\theta = 2^{n-1}\cot 2^{n-1}\theta - 2^n\cot 2^n\theta;$$

$\therefore$ by addition, $S = \cot\theta - 2^n\cot 2^n\theta.$

8. $$S = \frac{1}{2\cos\alpha\cos 2\alpha} + \frac{1}{2\cos 2\alpha\cos 3\alpha} + \frac{1}{2\cos 3\alpha\cos 4\alpha} + \ldots$$
$$= \frac{1}{2}\{\sec\alpha\sec 2\alpha + \sec 2\alpha\sec 3\alpha + \sec 3\alpha\sec 4\alpha + \ldots\}$$
$$= \frac{1}{2}\operatorname{cosec}\alpha\{\tan(n+1)\alpha - \tan\alpha\}.$$ [XXIII. a. Ex. 15.]

9. $$\sin^2\theta\sin 2\theta = \frac{\sin 2\theta}{2}(1 - \cos 2\theta);$$
$$\therefore \sin^2\theta\sin 2\theta = \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{4}.$$

Replacing θ by 2θ, we obtain

$$\frac{1}{2}\sin^2 2\theta\sin 4\theta = \frac{\sin 4\theta}{4} - \frac{\sin 8\theta}{8}.$$

Similarly, $$\frac{1}{4}\sin^2 4\theta\sin 8\theta = \frac{\sin 8\theta}{8} - \frac{\sin 16\theta}{16};$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{2^{n-1}}\sin^2 2^{n-1}\theta\sin 2^n\theta = \frac{\sin 2^n\theta}{2^n} - \frac{\sin 2^{n+1}\theta}{2^{n+1}};$$

$\therefore$ by addition, $$S = \frac{\sin 2\theta}{2} - \frac{\sin 2^{n+1}\theta}{2^{n+1}}.$$

10. $2\cos\theta\sin^2\frac{\theta}{2}=\cos\theta(1-\cos\theta)=1-\cos^2\theta-(1-\cos\theta);$

$$\therefore\ 2\cos\theta\sin^2\frac{\theta}{2}=\sin^2\theta-2\sin^2\frac{\theta}{2}.$$

Replacing θ by $\frac{\theta}{2}$, we obtain

$$2^2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2^2}=2\sin^2\frac{\theta}{2}-2^2\sin^2\frac{\theta}{2^2}.$$

Similarly, $$2^3\cos\frac{\theta}{2^2}\sin^2\frac{\theta}{2^3}=2^2\sin^2\frac{\theta}{2^2}-2^3\sin^2\frac{\theta}{2^3};$$

.....

$$2^n\cos\frac{\theta}{2^{n-1}}\sin^2\frac{\theta}{2^n}=2^{n-1}\sin^2\frac{\theta}{2^{n-1}}-2^n\sin^2\frac{\theta}{2^n};$$

$\therefore$ by addition, $$S=\sin^2\theta-2^n\sin^2\frac{\theta}{2^n}.$$

11. We have

$$\tan^{-1}\frac{x}{n(n+1)+x^2}=\tan^{-1}\frac{\frac{x}{n}-\frac{x}{n+1}}{1+\frac{x^2}{n(n+1)}}=\tan^{-1}\frac{x}{n}-\tan^{-1}\frac{x}{n+1},$$

and hence $$\tan^{-1}\frac{x}{1\cdot2+x^2}=\tan^{-1}x-\tan^{-1}\frac{x}{2};$$

$$\tan^{-1}\frac{x}{2\cdot3+x^2}=\tan^{-1}\frac{x}{2}-\tan^{-1}\frac{x}{3};$$

.....

$$\tan^{-1}\frac{x}{n(n+1)+x^2}=\tan^{-1}\frac{x}{n}-\tan^{-1}\frac{x}{n+1};$$

$\therefore$ by addition, $$S=\tan^{-1}x-\tan^{-1}\frac{x}{n+1}.$$

12. $$\tan^{-1}\frac{1}{1+n+n^2}=\tan^{-1}\frac{(n+1)-n}{1+n(n+1)}$$
$$=\tan^{-1}(n+1)-\tan^{-1}n.$$

$$\therefore\ \tan^{-1}\frac{1}{1+1+1^2}=\tan^{-1}2-\tan^{-1}1;$$

$$\tan^{-1}\frac{1}{1+2+2^2}=\tan^{-1}3-\tan^{-1}2;$$

.....

$$\tan^{-1}\frac{1}{1+n+n^2}=\tan^{-1}(n+1)-\tan^{-1}n;$$

$$\therefore\ S=\tan^{-1}(n+1)-\tan^{-1}1=\tan^{-1}(n+1)-\frac{\pi}{4}.$$

13. $$\tan^{-1}\frac{2n}{2+n^2+n^4}=\tan^{-1}\frac{(1+n+n^2)-(1-n+n^2)}{1+(1+n+n^2)(1-n+n^2)}$$
$$=\tan^{-1}(1+n+n^2)-\tan^{-1}(1-n+n^2).$$

$$\therefore\ \tan^{-1}\frac{2}{2+1^2+1^4}=\tan^{-1}3-\tan^{-1}1;$$

$$\tan^{-1}\frac{4}{2+2^2+2^4}=\tan^{-1}6-\tan^{-1}3;$$

.....

$$\tan^{-1}\frac{2n}{2+n^2+n^4}=\tan^{-1}(1+n+n^2)-\tan^{-1}(1-n+n^2);$$

$$\therefore\ S=\tan^{-1}(1+n+n^2)-\tan^{-1}1$$
$$=\tan^{-1}(1+n+n^2)-\frac{\pi}{4}.$$

14. $$\tan^{-1}\frac{2n}{1-n^2+n^4}=\tan^{-1}\frac{n^2+n-(n^2-n)}{1+(n^2+n)(n^2-n)}$$
$$=\tan^{-1}(n^2+n)-\tan^2(n^2-n).$$

$$\therefore\ \tan^{-1}\frac{2}{1-1^2+1^4}=\tan^{-1}2-\tan^{-1}0;$$

$$\tan^{-1}\frac{4}{1-2^2+2^4}=\tan^{-1}6-\tan^{-1}2;$$

....

$$\tan^{-1}\frac{2n}{1-n^2+n^4}=\tan^{-1}(n^2+n)-\tan^2(n^2-n);$$

$$\therefore\ S=\tan^{-1}(n^2+n).$$

15. Let O be the point on the circumference of the circle, and P, Q, R... the vertices of the polygon beginning with that nearest to O. Let L be the other extremity of the diameter through O, and let $\angle OLP=\theta$;

then $$OP=2r\sin\theta,\ OQ=2r\sin\left(\theta+\frac{\pi}{n}\right),\ OR=2r\sin\left(\theta+\frac{2\pi}{n}\right),\ \ldots$$

$\therefore$ sum of the squares of the chords

$$=4r^2\left\{\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{n}\right)+\sin^2\left(\theta+\frac{2\pi}{n}\right)+\ldots \text{ to } n \text{ terms}\right\}$$
$$=2r^2\left\{n-\cos 2\theta-\cos 2\left(\theta+\frac{\pi}{n}\right)-\cos 2\left(\theta+\frac{2\pi}{n}\right)-\ldots\right\}$$
$$=2nr^2-2r^2\frac{\sin\pi}{\sin\frac{\pi}{n}}\cos\left(2\theta+(n-1)\frac{\pi}{n}\right)$$
$$=2nr^2.$$

16. Let O be the centre of the inscribed circle, and let $\angle PON_1 = \theta$, where ON_1 is the perpendicular from O parallel to PA_1; let $OP = x$;

then $PA_1 = r - x\cos\theta$, $PA_2 = r - x\cos\left(\theta + \frac{\pi}{n}\right)$,

$PA_3 = r - x\cos\left(\theta + \frac{2\pi}{n}\right)$, $PA_4 = r - x\cos\left(\theta + \frac{3\pi}{n}\right)$, ...

$$\therefore PA_1 + PA_3 + \ldots + PA_{2n-1}$$
$$= nr - x\left\{\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \ldots \text{ to } n \text{ terms}\right\}$$
$$= nr. \qquad \text{[Art. 297.]}$$

Similarly $PA_2 + PA_4 + \ldots + PA_{2n}$
$$= nr - x\left\{\cos\left(\theta + \frac{\pi}{n}\right) + \cos\left(\theta + \frac{3\pi}{n}\right) + \ldots \text{ to } n \text{ terms}\right\}$$
$$= nr.$$

17. Let Q be the other extremity of the diameter through P, and let $\angle PQA_1 = \theta$, and let r be the radius of the circle; then

$$PA_1 = 2r\sin\theta,\ PA_2 = 2r\sin\left(\theta + \frac{\pi}{2n+1}\right),\ PA_3 = 2r\sin\left(\theta + \frac{2\pi}{2n+1}\right), \ldots$$

$$\therefore PA_1 + PA_3 + \ldots + PA_{2n+1}$$
$$= 2r\left\{\sin\theta + \sin\left(\theta + \frac{2\pi}{2n+1}\right) + \sin\left(\theta + \frac{4\pi}{2n+1}\right) + \ldots \text{ to } n+1 \text{ terms}\right\}$$
$$= 2r\,\frac{\sin\frac{(n+1)\pi}{2n+1}}{\sin\frac{\pi}{2n+1}}\sin\left(\theta + \frac{n\pi}{2n+1}\right);$$

and $PA_2 + PA_4 + \ldots + PA_{2n}$
$$= 2r\left\{\sin\left(\theta + \frac{\pi}{2n+1}\right) + \sin\left(\theta + \frac{3\pi}{2n+1}\right) + \ldots \text{ to } n \text{ terms}\right\}$$
$$= 2r\,\frac{\sin\frac{n\pi}{2n+1}}{\sin\frac{\pi}{2n+1}}\sin\frac{1}{2}\left(2\theta + \frac{\pi}{2n+1} + \frac{(2n-1)\pi}{2n+1}\right)$$
$$= 2r\,\frac{\sin\frac{(n+1)\pi}{2n+1}}{\sin\frac{\pi}{2n+1}}\sin\left(\theta + \frac{n\pi}{2n+1}\right).$$

$$\therefore PA_1 + PA_3 + \ldots + PA_{2n+1} = PA_2 + PA_4 + \ldots + PA_{2n}.$$

18. Let $p_1, p_2, p_3, \ldots p_n$ be the perpendiculars; then as in Ex. 16, we have

$$p_1 = r - r\cos\theta, \quad p_2 = r - r\cos\left(\theta + \frac{2\pi}{n}\right), \quad p_3 = r - r\cos\left(\theta + \frac{4\pi}{n}\right), \ \ldots$$

(i) $\Sigma p^2 = \{r - r\cos\theta\}^2 + \left\{r - r\cos\left(\theta + \frac{2\pi}{n}\right)\right\}^2 + \ldots$ to n terms

$$= nr^2 - 2r^2\left\{\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \ldots\right\} + r^2\left\{\cos^2\theta + \cos^2\left(\theta + \frac{2\pi}{n}\right) + \ldots\right\}.$$

Now $$\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \ldots = 0; \qquad \text{[Art. 297.]}$$

$$\therefore \ \Sigma p^2 = nr^2 + \frac{r^2}{2}\left\{n + \cos 2\theta + \cos 2\left(\theta + \frac{2\pi}{n}\right) + \ldots\right\}$$

$$= nr^2 + \frac{nr^2}{2} = \frac{3nr^2}{2}.$$

(ii) $\Sigma p^3 = nr^3 - 3r^3\left\{\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \ldots \text{ to } n \text{ terms}\right\}$

$$+ \frac{3r^3}{2}\left\{n + \cos 2\theta + \cos 2\left(\theta + \frac{2\pi}{n}\right) + \ldots\right\}$$

$$- \frac{r^3}{4}\left\{3\cos\theta + 3\cos\left(\theta + \frac{2\pi}{n}\right) + \ldots + \cos 3\theta + \cos 3\left(\theta + \frac{3\pi}{n}\right) + \ldots\right\}$$

$$= nr^3 + \frac{3nr^3}{2} = \frac{5nr^3}{2},$$

since all the series of cosines vanish by Art. 297.

EXAMPLES. XXIV. a. PAGE 301.

1. We have

$$\frac{1}{a}\cos\alpha + \frac{1}{b}\sin\alpha = \frac{1}{c}, \quad \frac{1}{a}\cos\beta + \frac{1}{b}\sin\beta = \frac{1}{c};$$

and the required result follows at once by cross multiplication as in **Ex. 1**, p. 297.

4. $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \frac{c^2 - b^2}{a^2 + b^2} - \frac{c^2 - a^2}{a^2 + b^2},$$

[a] *in Example 2, page 297.*

5. $\cos^2\frac{\alpha-\beta}{2}=\frac{1+\cos(\alpha-\beta)}{2}=\frac{1+\cos\alpha\cos\beta+\sin\alpha\sin\beta}{2}$

$$=\frac{(a^2+b^2)+(c^2-b^2)+(c^2-a^2)}{2(a^2+b^2)}$$

$$=\frac{c^2}{a^2+b^2}.$$

6. $$\sin 2\alpha+\sin 2\beta=2\sin(\alpha+\beta)\cos(\alpha-\beta).$$

From Example 4, we find that $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$;

and $$\cos(\alpha-\beta)=\frac{c^2-b^2}{a^2+b^2}+\frac{c^2-a^2}{a^2+b^2}=\frac{2c^2-a^2-b^2}{a^2+b^2}.$$

7. $\sin^2\alpha+\sin^2\beta=(\sin\alpha+\sin\beta)^2-2\sin\alpha\sin\beta$

$$=\left(\frac{2bc}{a^2+b^2}\right)^2-\frac{2(c^2-a^2)}{a^2+b^2}.$$ [See p. 298.]

8. Here α and β are solutions of $a\cos\theta+b\sin\theta=c$; hence as in Example 4, we find

$$\cos(\alpha+\beta)=\frac{a^2-b^2}{a^2+b^2},\text{ and therefore }\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}.$$

Again, $\cot\alpha+\cot\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$

$$=\frac{2ab}{a^2+b^2}\div\frac{c^2-a^2}{a^2+b^2}.$$ [See p. 298.]

9. By squaring and adding, we have

$$2+2\cos(\theta-\phi)=a^2+b^2.$$

Again, $$\cos(\theta+\phi)=\frac{a^2-b^2}{a^2+b^2}.$$ [Ex. 4, p. 299.]

And $$2\cos\theta\cos\phi=\cos(\theta+\phi)+\cos(\theta-\phi);$$

$$\therefore\ 4\cos\theta\cos\phi=\frac{2(a^2-b^2)}{a^2+b^2}+a^2+b^2-2$$

$$=\frac{(a^2+b^2)^2-4b^2}{a^2+b^2}.$$

10. $\cos 2\theta+\cos 2\phi=2\cos(\theta+\phi)\cos(\theta-\phi)$

$$=\frac{a^2-b^2}{a^2+b^2}(a^2+b^2-2),$$

as in the preceding example.

11. $$\tan\theta+\tan\phi=\frac{\sin(\theta+\phi)}{\cos\theta\cos\phi}$$

$$=\frac{2ab}{a^2+b^2}\div\frac{(a^2+b^2)^2-4b^2}{4(a^2+b^2)}.$$

[See Ex. 4, p. 299 and Ex. 9 above.]

12. $$\tan\frac{\theta}{2}+\tan\frac{\phi}{2}=\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta}{2}\cos\frac{\phi}{2}}=\frac{2\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta+\phi}{2}+\cos\frac{\theta-\phi}{2}}.$$

On multiplying numerator and denominator by $2\cos\frac{\theta+\phi}{2}$, this last fraction becomes

$$\frac{2\sin(\theta+\phi)}{1+\cos(\theta+\phi)+\cos\theta+\cos\phi}.$$

By substituting for $\sin(\theta+\phi)$ and $\cos(\theta+\phi)$ the values found in **Ex. 4**, p. 299, we obtain

$$\frac{4ab}{a^2+b^2}\div\left(1+\frac{a^2-b^2}{a^2+b^2}+a\right),$$

which reduces to $$\frac{4b}{a^2+b^2+2a}.$$

13. The expression $=\sin^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$

$=-\cos(\alpha+\beta)\cos(\alpha-\beta)-\cos^2\gamma+\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}\cos\gamma$

$=-\{\cos(\alpha+\beta)-\cos\gamma\}\{\cos(\alpha-\beta)-\cos\gamma\}$

$=4\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}\sin\frac{\gamma+\alpha-\beta}{2}$;

from which the second part of the question easily follows.

14. The expression $=\sin^2\alpha+(\sin^2\beta-\sin^2\gamma)+2\sin\alpha\sin\beta\cos\gamma$

$=\sin^2\alpha+\sin(\beta+\gamma)\sin(\beta-\gamma)+\sin\alpha\{\sin(\beta+\gamma)+\sin(\beta-\gamma)\}$

$=\{\sin\alpha+\sin(\beta+\gamma)\}\{\sin\alpha+\sin(\beta-\gamma)\}$

$=4\sin\frac{\alpha+\beta+\gamma}{2}\cos\frac{\beta+\gamma-\alpha}{2}\sin\frac{\alpha+\beta-\gamma}{2}\cos\frac{\alpha-\beta+\gamma}{2}$.

15. The expression $= \sin^2\alpha - (\cos^2\beta - \sin^2\gamma) - 2\sin\alpha\sin\beta\sin\gamma$

$= \sin^2\alpha - \cos(\beta+\gamma)\cos(\beta-\gamma) - \sin\alpha\{\cos(\beta-\gamma) - \cos(\beta+\gamma)\}$

$= \{\sin\alpha + \cos(\beta+\gamma)\}\{\sin\alpha - \cos(\beta-\gamma)\}$

$= -\left\{\cos\left(\frac{\pi}{2}-\alpha\right) + \cos(\beta+\gamma)\right\}\left\{\cos\left(\frac{\pi}{2}+\alpha\right) + \cos(\beta-\gamma)\right\}$

$= -4\cos\left(\frac{\beta+\gamma-\alpha}{2}+\frac{\pi}{4}\right)\cos\left(\frac{\beta+\gamma+\alpha}{2}-\frac{\pi}{4}\right)\cos\left(\frac{\alpha+\beta-\gamma}{2}+\frac{\pi}{4}\right)$

$$\cos\left(\frac{\alpha-\beta+\gamma}{2}+\frac{\pi}{4}\right)$$

16. $$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\cos\alpha+\cos\beta-\cos\alpha\cos\beta}{1+\cos\alpha-\cos\beta-\cos\alpha\cos\beta}$$

$$= \frac{(1-\cos\alpha)(1+\cos\beta)}{(1+\cos\alpha)(1-\cos\beta)} = \tan^2\frac{\alpha}{2}\cot^2\frac{\beta}{2}.$$

17. $$\tan^2\theta = \frac{2\sin\alpha\sin\beta}{1+\cos(\alpha+\beta)} = \frac{2\sin\alpha\sin\beta}{1+\cos\alpha\cos\beta-\sin\alpha\sin\beta};$$

$$\therefore\ 1+\tan^2\theta = \frac{1+\cos\alpha\cos\beta+\sin\alpha\sin\beta}{1+\cos\alpha\cos\beta-\sin\alpha\sin\beta};$$

that is, $$\sec^2\theta = \frac{1+\cos(\alpha-\beta)}{1+\cos(\alpha+\beta)} = \frac{\cos^2\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}}.$$

Taking the positive value of the square root, we have

$$\cos\theta = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

$$\therefore\ \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}.$$

18. Here $\tan\theta = \dfrac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$;

$$\therefore \sec^2\theta = 1+\frac{\sin^2\alpha\cos^2\beta}{(\cos\alpha+\sin\beta)^2} = \frac{(\cos\alpha+\sin\beta)^2+(1-\cos^2\alpha)(1-\sin^2\beta)}{(\cos\alpha+\sin\beta)^2}$$

$$= \frac{1+2\cos\alpha\sin\beta+\cos^2\alpha\sin^2\beta}{(\cos\alpha+\sin\beta)^2}.$$

Taking the positive value of the square root, we have

$$\cos\theta = \frac{\cos\alpha+\sin\beta}{1+\cos\alpha\sin\beta}.$$

$$\therefore \frac{1-\cos\theta}{1+\cos\theta} = \frac{(1-\cos\alpha)(1-\sin\beta)}{(1+\cos\alpha)(1+\sin\beta)}$$

$$= \frac{(1-\cos\alpha)\left\{1-\cos\left(\frac{\pi}{2}-\beta\right)\right\}}{(1+\cos\alpha)\left\{1+\cos\left(\frac{\pi}{2}-\beta\right)\right\}};$$

$$\therefore \tan^2\frac{\theta}{2} = \tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

19. This identity will be established if we shew that

$$\Sigma\sin^3 A\sin B\sin C = 2\sin A\sin B\sin C(1+\cos A\cos B\cos C),$$

that is, if $$\Sigma\sin^2 A = 2(1+\cos A\cos B\cos C).$$

Now $$\sin^2 A+\sin^2 B+\sin^2 C = \frac{1}{2}(3-\cos 2A-\cos 2B-\cos 2C)$$

$$= \frac{1}{2}(4+4\cos A\cos B\cos C).$$

[See XII. d. Ex. 9.]

20. Expressing a, b, c in terms of R, this identity will be proved if we shew that

$$\Sigma\sin A\cos^3 A = \Pi\sin A(1-4\cos A\cos B\cos C).$$

Now $$8\Sigma\sin A\cos^3 A = 4\Sigma\cos^2 A\sin 2A$$

$$= 2\Sigma(1+\cos 2A)\sin 2A$$

$$= 2\Sigma\sin 2A+\Sigma\sin 4A.$$

Now $$\Sigma\sin 2A = 4\Pi\sin A,$$

and $$\Sigma\sin 4A = -4\Pi\sin 2A;$$ [Ex. 7, p. 301];

$$\therefore \Sigma\sin A\cos^3 A = \Pi\sin A-4\Pi\sin A\cos A$$

$$= \Pi\sin A(1-4\cos A\cos B\cos C).$$

21. $\Sigma a^3 \cos(B-C) = 2R\Sigma a^2 \sin A \cos(B-C)$

$= 2R\Sigma a^2 \sin(B+C)\cos(B-C)$

$= R\Sigma a^2 (\sin 2B + \sin 2C)$.

Now $a^2 \sin 2B + b^2 \sin 2A = 2a \sin B \,.\, a\cos B + 2b \sin A \,.\, b\cos A$

$= 2a \sin B (a\cos B + b\cos A)$

$= 2ac \sin B = 4\Delta$.

Hence $\Sigma a^3 \cos(B-C) = 12R\Delta = 3abc$.

22. (1) We have, from the given equation,

$$(b\sin\theta - c)^2 = a^2\cos^2\theta = a^2(1-\sin^2\theta),$$

or $$(a^2+b^2)\sin^2\theta - 2bc\sin\theta + (c^2-a^2) = 0;$$

and this is the required equation since by hypothesis it is satisfied by $\sin\alpha$ and $\sin\beta$.

(2) The required equation is

$$x^2 - (\cos 2\alpha + \cos 2\beta)\, x + \cos 2\alpha \cos 2\beta = 0.$$

Now $\cos 2\alpha + \cos 2\beta = 2\cos(\alpha+\beta)\cos(\alpha-\beta)$

$$= \frac{2(a^2-b^2)}{a^2+b^2} \cdot \frac{2c^2-a^2-b^2}{a^2+b^2}.$$

[See solutions to examples 4 and 6 above.]

Again, $\cos 2\alpha \cos 2\beta = \cos^2(\alpha-\beta) - \sin^2(\alpha+\beta)$

$$= \frac{(2c^2-a^2-b^2)^2 - 4a^2b^2}{(a^2+b^2)^2}.$$

Hence, by substitution the required equation is obtained.

Otherwise. Let $\cos\alpha = y$, then the given equation may be written

$$(ay-c)^2 = b^2(1-y^2),$$

or $$(a^2+b^2)y^2 - b^2 + c^2 = 2acy \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If $$x = \cos 2\alpha = 2\cos^2\alpha - 1 = 2y^2 - 1,$$

we have $$y^2 = \frac{x+1}{2}.$$

Substituting in (1) we obtain the equation

$$\left\{(a^2+b^2)\frac{x+1}{2} - b^2 + c^2\right\}^2 = 4a^2c^2 \cdot \left(\frac{x+1}{2}\right),$$

which reduces to

$$(a^2+b^2)^2 x^2 - 2(a^2-b^2)(2c^2-a^2-b^2)\, x + (a^4+b^4+4c^4-2a^2b^2-4a^2c^2-4b^2c^2) = 0.$$

EXAMPLES. XXIV. b. PAGE 307.

1. $\Sigma \sin(\alpha-\theta)\sin(\beta-\gamma) = \Sigma(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\sin(\beta-\gamma)$

$= \cos\theta\,\Sigma \sin\alpha\sin(\beta-\gamma) - \sin\theta\,\Sigma\cos\alpha\sin(\beta-\gamma)$

$= 0.$

2. $\Sigma(\cos\beta\cos\gamma - \sin\beta\sin\gamma)\sin(\beta-\gamma) = \Sigma\cos(\beta+\gamma)\sin(\beta-\gamma)$

$= \frac{1}{2}\Sigma(\sin 2\beta - \sin 2\gamma)$

$= 0.$

$\therefore\ \Sigma\cos\beta\cos\gamma\sin(\beta-\gamma) = \Sigma\sin\beta\sin\gamma\sin(\beta-\gamma).$

3. $\Sigma\sin(\beta-\gamma)\cos(\beta+\gamma+\theta)$

$= \Sigma\sin(\beta-\gamma)\{\cos\theta\cos(\beta+\gamma) - \sin\theta\sin(\beta+\gamma)\}$

$= \cos\theta\,\Sigma\sin(\beta-\gamma)\cos(\beta+\gamma) - \sin\theta\,\Sigma\sin(\beta-\gamma)\sin(\beta+\gamma)$

$= \frac{1}{2}\cos\theta\,\Sigma(\sin 2\beta - \sin 2\gamma) - \frac{1}{2}\sin\theta\,\Sigma(\cos 2\gamma - \cos 2\beta)$

$= 0.$

4. $\cos 2(\beta-\gamma) + \cos 2(\gamma-\alpha) + \cos 2(\alpha-\beta)$

$= 2\cos(\beta-\alpha)\cos(\alpha+\beta-2\gamma) + 2\cos^2(\alpha-\beta) - 1$

$= 2\cos(\alpha-\beta)\{\cos(\alpha+\beta-2\gamma) + \cos(\alpha-\beta)\} - 1$

$= 4\cos(\alpha-\beta)\cos(\alpha-\gamma)\cos(\beta-\gamma) - 1.$

5. $\Sigma\sin\beta\sin\gamma\sin(\beta-\gamma)$

$= \frac{1}{2}\Sigma\{\cos(\beta-\gamma) - \cos(\beta+\gamma)\}\sin(\beta-\gamma)$

$= \frac{1}{4}\Sigma\sin 2(\beta-\gamma) - \frac{1}{4}\Sigma(\sin 2\beta - \sin 2\gamma)$

$= -\Pi\sin(\beta-\gamma).$ [Art. 306, Ex. 2.]

6. $\cot(\alpha-\beta) = \cot\{(\alpha-\gamma)-(\beta-\gamma)\}$

$= \dfrac{\cot(\alpha-\gamma)\cot(\beta-\gamma)+1}{\cot(\beta-\gamma) - \cot(\alpha-\gamma)}$;

by multiplying up and transposing we obtain the required result.

7. $2\Sigma \sin 3\alpha \sin(\beta-\gamma) = \cos(3\alpha-\beta+\gamma) - \cos(3\alpha+\beta-\gamma) + \cos(3\beta-\gamma+\alpha)$
$- \cos(3\beta+\gamma-\alpha) + \cos(3\gamma-\alpha+\beta) - \cos(3\gamma+\alpha-\beta).$

Combining the first and fourth terms, the second and fifth terms, the third and sixth terms, and dividing by 2, we obtain

$\Sigma \sin 3\alpha \sin(\beta-\gamma) = \sin(\alpha+\beta+\gamma)\{\sin 2(\beta-\alpha) + \sin 2(\alpha-\gamma) + \sin 2(\gamma-\beta)\}$
$= 4 \sin(\alpha+\beta+\gamma)\, \Pi \sin(\beta-\gamma).$ [Art. 306, Ex. 2.]

8. $4\Sigma \cos^3\alpha \sin(\beta-\gamma) = \Sigma(\cos 3\alpha + 3\cos\alpha)\sin(\beta-\gamma)$
$= \Sigma \cos 3\alpha \sin(\beta-\gamma)$
$= 4 \cos(\alpha+\beta+\gamma)\, \Pi \sin(\beta-\gamma).$ [Art. 306, Ex. 4.]

9. $4\Sigma \cos(\theta+\alpha)\sin(\theta-\alpha)\cos(\beta+\gamma)\sin(\beta-\gamma)$
$= \Sigma(\sin 2\theta - \sin 2\alpha)(\sin 2\beta - \sin 2\gamma)$
$= \sin 2\theta \Sigma(\sin 2\beta - \sin 2\gamma) - \Sigma \sin 2\alpha (\sin 2\beta - \sin 2\gamma)$
$= 0.$

10. In the identity $\Sigma bc(b-c) = -\Pi(b-c)$, put

$$a = \sin^2\alpha, \quad b = \sin^2\beta, \quad c = \sin^2\gamma;$$

then $$b - c = \sin^2\beta - \sin^2\gamma = \sin(\beta+\gamma)\sin(\beta-\gamma);$$

$$\therefore \Sigma \sin^2\beta \sin^2\gamma \sin(\beta+\gamma)\sin(\beta-\gamma) = -\Pi \sin(\beta+\gamma) \cdot \Pi \sin(\beta-\gamma).$$

11. In the identity $\Sigma bc(b-c) = -\Pi(b-c)$, put

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

then $$b - c = \cos 2\beta - \cos 2\gamma = -2 \sin(\beta+\gamma)\sin(\beta-\gamma);$$

$$\therefore -2\Sigma \cos 2\beta \cos 2\gamma \sin(\beta+\gamma)\sin(\beta-\gamma) = 8\Pi \sin(\beta+\gamma) \cdot \Pi \sin(\beta-\gamma).$$

12. In the identity $\Sigma a^2(b-c) = -\Pi(b-c)$, put

$$a = \cos 2\alpha, \quad b = \cos 2\beta, \quad c = \cos 2\gamma;$$

then $$b - c = \cos 2\beta - \cos 2\gamma = -2 \sin(\beta+\gamma)\sin(\beta-\gamma);$$

$$\therefore \Sigma\, 2\cos^2 2\alpha \sin(\beta+\gamma)\sin(\beta-\gamma) = -8\Pi \sin(\beta+\gamma) \cdot \Pi \sin(\beta-\gamma).$$

Also $$\Sigma \sin(\beta+\gamma)\sin(\beta-\gamma) = 0;$$

whence by subtraction, we have

$$\Sigma(2\cos^2 2\alpha - 1)\sin(\beta+\gamma)\sin(\beta-\gamma) = -8\Pi \sin(\beta+\gamma) \cdot \Pi \sin(\beta-\gamma).$$

13. In the identity $\Sigma a^3(b-c) = -(a+b+c)\,\Pi(b-c)$, put

$$a=\sin\alpha, \quad b=\sin\beta, \quad c=\sin\gamma;$$

$$\therefore\ \Sigma \sin^3\alpha(\sin\beta-\sin\gamma) = -(\sin\alpha+\sin\beta+\sin\gamma)\,\Pi(\sin\beta-\sin\gamma)\ldots(1).$$

But $$\Sigma \sin\alpha(\sin\beta-\sin\gamma)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

multiply (2) by 3, and (1) by 4; then by subtraction we obtain

$$\Sigma(3\sin\alpha-4\sin^3\alpha)(\sin\beta-\sin\gamma) = 4(\sin\alpha+\sin\beta+\sin\gamma)\,\Pi(\sin\beta-\sin\gamma).$$

14. If $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

The condition $a+b+c=0$ is satisfied, if $a=\sin(\beta+\gamma)\sin(\beta-\gamma)$, and b and c are equal to corresponding quantities.

15. The condition $a+b+c=0$ is satisfied, if $a=\cos(\beta+\gamma+\theta)\sin(\beta-\gamma)$, and b and c are equal to corresponding quantities.

16. We proceed exactly as in Art. 309, and shew that

$$\alpha+\beta+\gamma=n\pi;$$

$$\therefore\ 3\alpha+3\beta+3\gamma=3n\pi.$$

From this relation it is easy to shew that

$$\Sigma\tan 3\alpha = \Pi\tan 3\alpha;$$

$$\therefore\ \Sigma\frac{3x-x^3}{1-3x^2} = \Pi\frac{3x-x^3}{1-3x^2}.$$

17. Put $x=\cot\alpha$, $y=\cot\beta$, $z=\cot\gamma$; then

$$\cot\beta\cot\gamma+\cot\gamma\cot\alpha+\cot\alpha\cot\beta=1;$$

$$\therefore\ \cot\alpha = -\frac{\cot\beta\cot\gamma-1}{\cot\gamma+\cot\beta} = -\cot(\beta+\gamma);$$

$$\therefore\ \alpha=n\pi-(\beta+\gamma), \text{ or } \alpha+\beta+\gamma=n\pi;$$

$$\therefore\ 2\alpha+2\beta+2\gamma=2n\pi.$$

From this relation it is easy to shew that

$$\cot 2\beta\cot 2\gamma+\cot 2\gamma\cot 2\alpha+\cot 2\alpha\cot 2\beta=1;$$

$$\therefore\ \frac{(y^2-1)(z^2-1)}{4yz}+\frac{(z^2-1)(x^2-1)}{4zx}+\frac{(x^2-1)(y^2-1)}{4xy}=1,$$

$$\therefore\ \Sigma x(1-y^2)(1-z^2)=4xyz.$$

EXAMPLES. XXIV. c. PAGE 311.

1. If $$A+B+C=0,$$

then $$\cot C = -\cot(A+B) = -\frac{\cot A \cot B - 1}{\cot B + \cot A};$$

that is, $$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

The given condition is satisfied if

$$A = 2\beta + \gamma - 3\alpha, \quad B = 2\gamma + \alpha - 3\beta, \quad C = 2\alpha + \beta - 3\gamma.$$

2. (1) In Example 4, Art. 133, we have proved that

$$4 \sin\alpha \sin\beta \sin\gamma = \Sigma \sin(\beta+\gamma-\alpha) - \sin(\alpha+\beta+\gamma).$$

In this identity first replace α, β, γ by $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ respectively, and secondly replace α, β, γ by 2α, 2β, 2γ respectively.

Thus $$8\Pi \sin(\beta+\gamma) = 2\sin 2\alpha - 2\sin 2(\alpha+\beta+\gamma),$$

and $$4\Pi \sin 2\alpha = \Sigma \sin 2(\beta+\gamma-\alpha) - \sin 2(\alpha+\beta+\gamma).$$

$\therefore$ $$\begin{aligned} &8\Pi \sin(\beta+\gamma) + 4\Pi \sin 2\alpha \\ &= 2\Sigma \sin 2\alpha + \Sigma \sin 2(\beta+\gamma-\alpha) - 3\sin 2(\alpha+\beta+\gamma) \\ &= 2\Sigma \sin 2\alpha + \Sigma\{\sin 2(\beta+\gamma-\alpha) - \sin 2(\alpha+\beta+\gamma)\} \\ &= 2\Sigma \sin 2\alpha - 2\Sigma \cos 2(\beta+\gamma) \sin 2\alpha \\ &= 2\Sigma \sin 2\alpha \{1 - \cos 2(\beta+\gamma)\} \\ &= 4\Sigma \sin 2\alpha \sin^2(\beta+\gamma). \end{aligned}$$

(2) In the first part of this Example, we have seen that

$$8\Pi \sin\alpha = 2\Sigma \sin(\beta+\gamma-\alpha) - 2\sin(\alpha+\beta+\gamma).$$

By replacing α, β, γ by $\beta+\gamma-\alpha$, $\gamma+\alpha-\beta$, $\alpha+\beta-\gamma$ respectively, we have

$$4\Pi \sin(\beta+\gamma-\alpha) = \Sigma \sin(3\alpha-\beta-\gamma) - \sin(\alpha+\beta+\gamma).$$

$\therefore$ $$\begin{aligned} &4\Pi \sin(\beta+\gamma-\alpha) + 8\Pi \sin\alpha \\ &= 2\Sigma \sin(\beta+\gamma-\alpha) + \Sigma \sin(3\alpha-\beta-\gamma) - 3\sin(\alpha+\beta+\gamma) \\ &= 2\Sigma \sin(\beta+\gamma-\alpha) + \Sigma\{\sin(3\alpha-\beta-\gamma) - \sin(\alpha+\beta+\gamma)\} \\ &= 2\Sigma \sin(\beta+\gamma-\alpha) - 2\Sigma \cos 2\alpha \sin(\beta+\gamma-\alpha) \\ &= 2\Sigma \sin(\beta+\gamma-\alpha)\{1 - \cos 2\alpha\} \\ &= 4\Sigma \sin^2\alpha \sin(\beta+\gamma-\alpha). \end{aligned}$$

3. (1) This is equivalent to proving that in the pedal triangle

$$a'^2 - b'^2 = 2R'c' \sin (A' - B').$$

Now in *any* triangle

$$\begin{aligned} a^2 - b^2 &= 4R^2(\sin^2 A - \sin^2 B) \\ &= 4R^2 \sin (A+B) \sin (A-B) \\ &= 2R \,.\, 2R \sin C \,.\, \sin (A-B) \\ &= 2Rc \sin (A-B). \end{aligned}$$

(2) This is equivalent to proving that in the ex-central triangle

$$a_1^2 - b_1^2 = 2R_1 c_1 \sin (A_1 - B_1).$$

This identity has been proved in (1).

(3) This is equivalent to proving that in the pedal triangle

$$\Sigma (b'+c') \tan \frac{A'}{2} = 4R' \Sigma \cos A'.$$

Now in *any* triangle

$$\begin{aligned} \Sigma (b+c) \tan \frac{A}{2} &= 2R\Sigma (\sin B + \sin C) \tan \frac{A}{2} \\ &= 4R\Sigma \sin \frac{B+C}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \div \cos \frac{A}{2} \\ &= 4R\Sigma \cos \frac{B-C}{2} \cos \frac{B+C}{2} \\ &= 2R\Sigma (\cos B + \cos C) \\ &= 4R (\cos A + \cos B + \cos C). \end{aligned}$$

4. We have $\sin^2 2\theta = 4 \sin^2 \alpha \sin^2 \gamma = (1 - \cos 2\alpha)(1 - \cos 2\gamma)$;

$$\therefore 1 - \cos^2 2\theta = \left(1 - \frac{\cos 2\theta}{\cos 2\beta}\right)\left(1 - \frac{\cos 2\theta}{\cos 2\delta}\right);$$

$$\therefore \cos 2\theta \left(\frac{1}{\cos 2\beta} + \frac{1}{\cos 2\delta}\right) = \cos^2 2\theta \left(1 + \frac{1}{\cos 2\beta \cos 2\delta}\right);$$

$$\therefore \cos 2\theta = \frac{\cos 2\beta + \cos 2\delta}{1 + \cos 2\beta \cos 2\delta};$$

$$\therefore \frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \frac{(1 - \cos 2\beta)(1 - \cos 2\delta)}{(1 + \cos 2\beta)(1 + \cos 2\delta)};$$

$$\therefore \tan^2 \theta = \tan^2 \beta \tan^2 \delta.$$

5. We have $\tan^2\frac{\gamma}{2}=\tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}$;

$$\therefore \frac{1-\cos\gamma}{1+\cos\gamma}=\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\phi}{1+\cos\phi}$$

$$=\frac{1-\cos\alpha\cos\gamma}{1+\cos\alpha\cos\gamma}\cdot\frac{1-\cos\beta\cos\gamma}{1+\cos\beta\cos\gamma}.$$

Componendo and Dividendo,

$$\frac{1}{\cos\gamma}=\frac{1+\cos\alpha\cos\beta\cos^2\gamma}{(\cos\alpha+\cos\beta)\cos\gamma};$$

$$\therefore \cos^2\gamma=\frac{\cos\alpha+\cos\beta-1}{\cos\alpha\cos\beta};$$

$$\therefore \sin^2\gamma=\frac{(1-\cos\alpha)(1-\cos\beta)}{\cos\alpha\cos\beta}=(\sec\alpha-1)(\sec\beta-1).$$

6. By solving for $\cos\theta$, we have

$$\cos\theta=\frac{\sin^2\beta\cos^2\alpha-\sin^2\alpha\cos^2\beta}{\sin^2\beta\cos\alpha-\sin^2\alpha\cos\beta}.$$

By substituting $1-\cos^2\beta$ for $\sin^2\beta$, and $1-\cos^2\alpha$ for $\sin^2\alpha$, we have

$$\cos\theta=\frac{\cos^2\alpha-\cos^2\beta}{(1-\cos^2\beta)\cos\alpha-(1-\cos^2\alpha)\cos\beta}$$

$$=\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}.$$

$$\therefore \frac{1-\cos\theta}{1+\cos\theta}=\frac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)};$$

$$\therefore \tan^2\frac{\theta}{2}=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}.$$

7. From the two given equations, we have

$$\sec\alpha=\frac{\tan\beta}{\tan\theta},\text{ and }\tan\alpha=\frac{\tan\gamma}{\sin\theta};$$

$$\therefore \frac{\tan^2\beta}{\tan^2\theta}-\frac{\tan^2\gamma}{\sin^2\theta}=1;$$

$$\therefore \frac{\tan^2\beta\cos^2\theta}{1-\cos^2\theta}-\frac{\tan^2\gamma}{1-\cos^2\theta}=1;$$

$$\therefore \tan^2\beta\cos^2\theta-\tan^2\gamma=1-\cos^2\theta;$$

$$\therefore \cos^2\theta=\frac{1+\tan^2\gamma}{1+\tan^2\beta}=\frac{\sec^2\gamma}{\sec^2\beta}.$$

8. We have $bc\cos\alpha\cos\phi + ac\sin\alpha\sin\phi - ab = 0$,

and $bc\cos\beta\cos\phi + ac\sin\beta\sin\phi - ab = 0$;

whence by cross multiplication

$$\frac{\cos\phi}{a^2bc(\sin\beta-\sin\alpha)} = \frac{\sin\phi}{ab^2c(\cos\alpha-\cos\beta)} = \frac{1}{abc^2\sin(\beta-\alpha)};$$

$$\therefore \frac{\cos\phi}{a\cos\frac{\alpha+\beta}{2}} = \frac{\sin\phi}{b\sin\frac{\alpha+\beta}{2}} = \frac{1}{c\cos\frac{\alpha-\beta}{2}};$$

$$\therefore a^2\cos^2\frac{\alpha+\beta}{2} + b^2\sin^2\frac{\alpha+\beta}{2} = c^2\cos^2\frac{\alpha-\beta}{2};$$

$$\therefore a^2\{1+\cos(\alpha+\beta)\} + b^2\{1-\cos(\alpha+\beta)\} = c^2\{1+\cos(\alpha-\beta)\};$$

$$\therefore (b^2+c^2-a^2)\cos\alpha\cos\beta + (c^2+a^2-b^2)\sin\alpha\sin\beta = a^2+b^2-c^2.$$

9. From the given equation, we have

$$\sin^2\alpha(\cos\theta-\cos\alpha)^2 = \cos^2\alpha\sin^2\theta = \cos^2\alpha(1-\cos^2\theta);$$

$$\therefore \cos^2\theta - 2\cos\alpha\sin^2\alpha\cos\theta - \cos^4\alpha = 0;$$

which is a quadratic in $\cos\theta$ with roots $\cos\beta$ and $\cos\gamma$.

$$\therefore \cos\beta\cos\gamma = \cos^4\alpha.$$

Similarly, from the equation

$$\cos^2\alpha(\sin\theta-\sin\alpha)^2 = \sin^2\alpha\cos^2\theta = \sin^2\alpha(1-\sin^2\theta),$$

we may shew that $\sin\beta\sin\gamma = \sin^4\alpha$.

$$\therefore \frac{\cos\beta\cos\gamma}{\cos^2\alpha} + \frac{\sin\beta\sin\gamma}{\sin^2\alpha} = \cos^2\alpha + \sin^2\alpha = 1.$$

10. We have $k^2\cos\beta\cos\alpha + k\sin\alpha + (k\sin\beta+1) = 0$ .....(1),

and $k^2\cos\gamma\cos\alpha + k\sin\alpha + (k\sin\gamma+1) = 0$ .....(2);

whence by cross multiplication,

$$\frac{\cos\alpha}{k^2(\sin\beta-\sin\gamma)} = \frac{\sin\alpha}{k^3\sin(\beta-\gamma)+k^2(\cos\gamma-\cos\beta)} = \frac{1}{k^3(\cos\beta-\cos\gamma)};$$

$$\therefore \frac{\cos\alpha}{\cos\frac{\beta+\gamma}{2}} = \frac{\sin\alpha}{k\cos\frac{\beta-\gamma}{2}+\sin\frac{\beta+\gamma}{2}} = -\frac{1}{k\sin\frac{\beta+\gamma}{2}};$$

$$\therefore \cos^2\frac{\beta+\gamma}{2} + \left\{k\cos\frac{\beta-\gamma}{2}+\sin\frac{\beta+\gamma}{2}\right\}^2 = k^2\sin^2\frac{\beta+\gamma}{2};$$

$$\therefore k^2\left(\cos^2\frac{\beta-\gamma}{2}-\sin^2\frac{\beta+\gamma}{2}\right) + k(\sin\beta+\sin\gamma)+1 = 0;$$

$$\therefore k^2\cos\beta\cos\gamma + k(\sin\beta+\sin\gamma)+1 = 0.$$

Otherwise. Form a quadratic equation in $\sin\theta$; thus

$$k^4\cos^2\alpha\,(1-\sin^2\theta)=(1+k\sin\alpha+k\sin\theta)^2;$$

$\sin\beta+\sin\gamma=$ sum of roots of this equation

$$=-\frac{\text{coefficient of }\sin\theta}{\text{coefficient of }\sin^2\theta}$$

$$=-\frac{2\,(1+k\sin\alpha)\,k}{k^2\,(1+k^2\cos^2\alpha)};$$

$$\therefore\ k\,(\sin\beta+\sin\gamma)=-\frac{2\,(1+k\sin\alpha)}{1+k^2\cos^2\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Again, form a quadratic equation in $\cos\theta$; thus

$$k^2\,(1-\cos^2\theta)=(1+k\sin\alpha+k^2\cos\alpha\cos\theta)^2;$$

$\cos\beta\cos\gamma=$ product of roots of this equation

$$=\frac{k^2-(1+k\sin\alpha)^2}{-k^2-k^4\cos^2\alpha};$$

$$\therefore\ k^2\cos\beta\cos\gamma=\frac{1+2k\sin\alpha-k^2\cos^2\alpha}{1+k^2\cos^2\alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

By adding (1) and (2) we obtain the required result.

11. We have

$$\sin^2\alpha\cos\beta\cos\phi+\cos^2\alpha\sin\beta\sin\phi+\cos^2\alpha\sin^2\alpha=0,$$

and $$\sin^2\alpha\cos\gamma\cos\phi+\cos^2\alpha\sin\gamma\sin\phi+\cos^2\alpha\sin^2\alpha=0;$$

whence by cross multiplication

$$\frac{\cos\phi}{\cos^4\alpha\sin^2\alpha\,(\sin\beta-\sin\gamma)}=\frac{\sin\phi}{\cos^2\alpha\sin^4\alpha\,(\cos\gamma-\cos\beta)}$$

$$=\frac{1}{\sin^2\alpha\cos^2\alpha\sin(\gamma-\beta)};$$

$$\therefore\ \frac{\cos\phi}{\cos^2\alpha\cos\dfrac{\beta+\gamma}{2}}=\frac{\sin\phi}{\sin^2\alpha\sin\dfrac{\beta+\gamma}{2}}=-\frac{1}{\cos\dfrac{\beta-\gamma}{2}};$$

$$\therefore\ \cos^4\alpha\cos^2\frac{\beta+\gamma}{2}+\sin^4\alpha\sin^2\frac{\beta+\gamma}{2}=\cos^2\frac{\beta-\gamma}{2};$$

$$\therefore\ \cos^4\alpha\,\{1+\cos(\beta+\gamma)\}+\sin^4\alpha\,\{1-\cos(\beta+\gamma)\}=1+\cos(\beta-\gamma);$$

$$\therefore\ \cos\beta\cos\gamma\,(1-\cos^4\alpha+\sin^4\alpha)+\sin\beta\sin\gamma\,(1+\cos^4\alpha-\sin^4\alpha)$$

$$=\cos^4\alpha+\sin^4\alpha-1.$$

Writing $\cos^4\alpha+\sin^4\alpha+2\sin^2\alpha\cos^2\alpha$ instead of unity we have

$$2\cos\beta\cos\gamma\sin^2\alpha\,(\cos^2\alpha+\sin^2\alpha)+2\sin\beta\sin\gamma\cos^2\alpha\,(\cos^2\alpha+\sin^2\alpha)$$

$$=-2\sin^2\alpha\cos^2\alpha;$$

that is, $$\sin^2\alpha\cos\beta\cos\gamma+\cos^2\alpha\sin\beta\sin\gamma+\sin^2\alpha\cos^2\alpha=0.$$

EXAMPLES. XXV. a. PAGE 318.

1. $p\cot\theta+q\tan\theta=(\sqrt{p\cot\theta}-\sqrt{q\tan\theta})^2+2\sqrt{pq}$.

2. $4\sin^2\theta+\operatorname{cosec}^2\theta=(2\sin\theta-\operatorname{cosec}\theta)^2+4$.

3. $8\sec^2\theta+18\cos^2\theta=2\{(2\sec\theta-3\cos\theta)^2+12\}$.

4. $3-2\cos\theta+\cos^2\theta=2+(1-\cos\theta)^2$.

5. We have $\tan^2\beta+\tan^2\gamma>2\tan\beta\tan\gamma$, and two corresponding inequalities. [See Art. 316.]

6. Since $(1-\sin\alpha)^2$ is positive, $1+\sin^2\alpha>2\sin\alpha$.

Similarly, $1+\sin^2\beta>2\sin\beta$.

$\therefore\ 2+\sin^2\alpha+\sin^2\beta>2\sin\alpha+2\sin\beta$.

7. $\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$.

8. $\cos\theta+\sqrt{3}\sin\theta=2\sin\left(\theta+\frac{\pi}{6}\right)$.

9. $a\cos(\alpha+\theta)+b\sin\theta=a\cos\alpha\cos\theta+(b-a\sin\alpha)\sin\theta$.

$\therefore$ maximum value $=\sqrt{a^2\cos^2\alpha+(b-a\sin\alpha)^2}$. [Art. 317.]

10. $p\cos\theta+q\sin(\alpha+\theta)=(p+q\sin\alpha)\cos\theta+q\cos\alpha\sin\theta$.

$\therefore$ maximum value $=\sqrt{(p+q\sin\alpha)^2+q^2\cos^2\alpha}$.

11. $\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2\sin\frac{\sigma}{2}\cos\frac{\alpha-\beta}{2}$.

$\therefore$ maximum value $=2\sin\frac{\sigma}{2}$.

12. $\sin\alpha\sin\beta=\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}=\frac{1}{2}\{\cos(\alpha-\beta)-\cos\sigma\}$.

$\therefore$ maximum value $=\frac{1}{2}(1-\cos\sigma)=\sin^2\frac{\sigma}{2}$.

13. $\tan\alpha+\tan\beta=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\frac{\sin\sigma}{\cos\alpha\cos\beta}$.

By Art. 319, the denominator is a maximum when $\alpha=\beta$, and in this *case* $\tan\alpha+\tan\beta$ is a minimum, its value being $2\tan\frac{\sigma}{2}$.

14. $\operatorname{cosec}\alpha+\operatorname{cosec}\beta=\dfrac{\sin\alpha+\sin\beta}{\sin\alpha\sin\beta}$

$$=\frac{4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}=\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha-\beta}{2}-\cos^2\frac{\alpha+\beta}{2}}$$

$$=\sin\frac{\alpha+\beta}{2}\left(\frac{1}{\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}}+\frac{1}{\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}}\right).$$

Since $\alpha+\beta$ is constant, this expression is least when the denominators are greatest, that is, when $\alpha=\beta=\frac{\sigma}{2}$.

Thus the minimum value is $2\operatorname{cosec}\frac{\sigma}{2}$.

15. $\cos A\cos B\cos C=\frac{1}{2}\cos C\{\cos(A-B)+\cos(A+B)\}$

$$=\frac{1}{2}\cos C\{\cos(A-B)-\cos C\}.$$

Supposing C constant, this expression is not a maximum unless $A=B$. Similarly, we may shew that the given expression is not a maximum unless $A=B=C=60°$. In this case its value is $\cos^3 60°$ or $\frac{1}{8}$.

16. $\cot A+\cot B+\cot C=\dfrac{\sin(A+B)}{\sin A\sin B}+\cot C=\dfrac{\sin C}{\sin A\sin B}+\cot C.$

Supposing C constant, $\sin A\sin B$ is a maximum when $A=B$, and in this case the given expression is a minimum. Thus the given expression is a minimum when $A=B=C$, and its value is $3\cot 60°$ or $\sqrt{3}$.

17. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=\frac{1}{2}(1-\cos A)+\ldots+\ldots$

$$=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C).$$

As in Example 1, p. 315, it is easy to shew that $\cos A+\cos B+\cos C$ is a maximum when $A=B=C$, and in this case the given expression is a minimum, its value being $3\sin^2\frac{60°}{2}$ or $\frac{3}{4}$.

18. If C is constant, $A+B$ is constant, and therefore $\sec A+\sec B$ is a minimum when $A=B$. [See Ex. 2, p. 316.]

Thus the given expression is a minimum when $A=B=C$, its value bein 3 sec 60° or 6.

19. $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}$

$$=\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2-2\Sigma\tan\frac{B}{2}\tan\frac{C}{2}$$

$$=\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2-2.$$

As in Example 16, it is easy to shew that $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}$ is a minimum when $A=B=C$, and in this case the given expression is a minimum, its value being $3\tan^2\frac{60^\circ}{2}$ or 1.

20. $\cot^2A+\cot^2B+\cot^2C=(\cot A+\cot B+\cot C)^2-2\Sigma\cot B\cot C$

$$=(\cot A+\cot B+\cot C)^2-2.$$

But it has been shewn in Example 16 that the right side is a minimum when $A=B=C$, and in this case the given expression is a minimum, its value being $3\cot^2 60^\circ$ or 1.

21. $2(a\sin^2\theta+b\sin\theta\cos\theta+c\cos^2\theta)$

$$=a(1-\cos 2\theta)+b\sin 2\theta+c(1+\cos 2\theta)$$

$$=a+c+b\sin 2\theta-(a-c)\cos 2\theta \quad \text{.....(1).}$$

The greatest value of $b\sin 2\theta-(a-c)\cos 2\theta$ is $\sqrt{b^2+(a-c)^2}$, which is less than $\sqrt{4ac+(a-c)^2}$ or $a+c$, since $b^2<4ac$.

Hence as in Art. 317, the maximum and minimum values of (1) are

$$a+c\pm\sqrt{b^2+(a-c)^2}.$$

22. $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\cos\frac{\alpha+\beta+2\gamma}{2}\sin\frac{\alpha+\beta}{2}$$

$$=4\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+\alpha}{2}\sin\frac{\beta+\gamma}{2}.$$

The expression on the right is positive, since each of its component factors is positive.

$$\therefore\ \sin\alpha+\sin\beta+\sin\gamma>\sin(\alpha+\beta+\gamma).$$

23. Let $x=a\operatorname{cosec}\theta-b\cot\theta$, and put $\cot\theta=t$;

then $$x=a\sqrt{1+t^2}-bt.$$

As in Ex. 2, page 317, we may shew that $x>\sqrt{a^2-b^2}$;

$$\therefore\ a\operatorname{cosec}\theta-b\cot\theta>\sqrt{a^2-b^2}.$$

24. Let $x=\dfrac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}=\dfrac{1-\tan\theta+\tan^2\theta}{1+\tan\theta+\tan^2\theta}$;

$$\therefore\ \tan^2\theta\,(x-1)+\tan\theta\,(x+1)+x-1=0.$$

In order that the values of $\tan\theta$ found from this quadratic may be real, we must have

$$(x+1)^2>4\,(x-1)^2;$$

that is, $-3x^2+10x-3$ must be positive;

that is, $(3x-1)\,(x-3)$ must be negative.

Hence x must lie between 3 and $\dfrac{1}{3}$.

25. Denote the expression by x; then

$$x=\frac{\tan^4\theta+\tan^2\theta-1}{\tan^4\theta-\tan^2\theta+1};$$

$$\therefore\ \tan^4\theta\,(x-1)-\tan^2\theta\,(x+1)+x+1=0.$$

In order that the values of $\tan^2\theta$ found from this equation may be real, we must have

$$(x+1)^2>4\,(x+1)\,(x-1);$$

$$\therefore\ (x+1)(5-3x)>0.$$

Thus the greatest value of x is $\dfrac{5}{3}$.

26. We have

$$(a^2+b^2+c^2)(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)-(a\cos\alpha+b\cos\beta+c\cos\gamma)^2$$
$$=(b\cos\gamma-c\cos\beta)^2+(c\cos\alpha-a\cos\gamma)^2+(a\cos\beta-b\cos\alpha)^2,$$

the minimum value of which is zero.

$$\therefore\ \text{minimum value of } (a^2+b^2+c^2)(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)-k^2=0.$$

Again, $(a+b+c)(a\cos^2\alpha+b\cos^2\beta+c\cos^2\gamma)-(a\cos\alpha+b\cos\beta+c\cos\gamma)^2$

$$=bc\,(\cos\beta-\cos\gamma)^2+ca\,(\cos\gamma-\cos\alpha)^2+ab\,(\cos\alpha-\cos\beta)^2,$$

the minimum value of which is zero.

$$\therefore\ \text{minimum value of } (a+b+c)(a\cos^2\alpha+b\cos^2\beta+c\cos^2\gamma)-k^2=$$

EXAMPLES. XXV. b. PAGE 324.

1. By squaring each of the given equations and adding, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

2. By transposition, we have

$$a\sec\theta=x\tan\theta+y,\quad b\sec\theta=x-y\tan\theta;$$

whence by squaring and adding,

$$(a^2+b^2)\sec^2\theta=(x^2+y^2)(1+\tan^2\theta),$$

or
$$x^2+y^2=a^2+b^2.$$

3. We have $\cos\theta+\sin\theta=a$, and $\cos^2\theta-\sin^2\theta=b$;

$$\therefore\ \cos\theta-\sin\theta=\frac{b}{a};$$

$$\therefore\ a^2+\frac{b^2}{a^2}=2.$$

4. We have $x^2=(\sin\theta+\cos\theta)^2=1+\sin 2\theta$,

and
$$y=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{2}{\sin 2\theta};$$

$$\therefore\ y(x^2-1)=2.$$

5. By addition and subtraction, we have

$$\cot\theta=\frac{a+b}{2},\ \text{and}\ \cos\theta=\frac{a-b}{2};$$

whence by division,
$$\sin\theta=\frac{a-b}{a+b};$$

$$\therefore\left(\frac{a-b}{2}\right)^2+\left(\frac{a-b}{a+b}\right)^2=1,\ \text{or}\ \left(\frac{a-b}{2}\right)^2=\frac{4ab}{(a+b)^2},$$

that is,
$$(a^2-b^2)^2=16ab.$$

6. Here $x=\cot\theta+\dfrac{1}{\cot\theta}=\dfrac{\cot^2\theta+1}{\cot\theta}=\dfrac{\operatorname{cosec}^2\theta}{\cot\theta}$,

and
$$y=\operatorname{cosec}\theta-\frac{1}{\operatorname{cosec}\theta}=\frac{\operatorname{cosec}^2\theta-1}{\operatorname{cosec}\theta}=\frac{\cot^2\theta}{\operatorname{cosec}\theta};$$

$$\therefore\ x^2y=\operatorname{cosec}^3\theta,\ \text{and}\ xy^2=\cot^3\theta.$$

But
$$\operatorname{cosec}^2\theta-\cot^2\theta=1;$$

$$\therefore\ x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{2}{3}}y^{\frac{4}{3}}=1.$$

7. Here $$a^3=\frac{1}{\sin\theta}-\sin\theta=\frac{\cos^2\theta}{\sin\theta},$$

and $$b^3=\frac{1}{\cos\theta}-\cos\theta=\frac{\sin^2\theta}{\cos\theta};$$

$$\therefore\ a^6b^3=\cos^3\theta,\text{ or } a^2b=\cos\theta;$$

and $$a^3b^6=\sin^3\theta,\text{ or } ab^2=\sin\theta;$$

$$\therefore\ a^4b^2+a^2b^4=1.$$

8. By substituting for $\cos 3\theta$ and $\sin 3\theta$, we have

$$4x=4a\cos^3\theta,\text{ or } x^{\frac{1}{3}}=a^{\frac{1}{3}}\cos\theta,$$

and $$4y=4a\sin^3\theta,\text{ or } y^{\frac{1}{3}}=a^{\frac{1}{3}}\sin\theta;$$

$$\therefore\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

9. By transposition, we have

$$x(1+\tan^2\theta)=a\tan^3\theta,\text{ or } x=\frac{a\tan^3\theta}{\sec^2\theta},$$

and $$y(\sec^2\theta-1)=a\sec^3\theta,\text{ or } y=\frac{a\sec^3\theta}{\tan^2\theta};$$

$$\therefore\ x^3y^2=a^5\tan^5\theta,\text{ and } x^2y^3=a^5\sec^5\theta.$$

$$\therefore\ (x^2y^3)^{\frac{2}{5}}-(x^3y^2)^{\frac{2}{5}}=a^2(\sec^2\theta-\tan^2\theta)=a^2.$$

10. Here $$x=a\cos\theta(2\cos 2\theta-1)=a\cos\theta(4\cos^2\theta-3)$$
$$=a\cos 3\theta.$$

Similarly $$y=b\sin 3\theta.$$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

11. Here $$\sin\theta\sin\alpha+\cos\theta\cos\alpha=a,$$

and $$\sin\theta\cos\beta-\cos\theta\sin\beta=b;$$

$$\therefore\ \sin\theta\cos(\alpha-\beta)=a\sin\beta+b\cos\alpha,$$

and $$\cos\theta\cos(\alpha-\beta)=a\cos\beta-b\sin\alpha;$$

$$\therefore\ \cos^2(\alpha-\beta)=a^2+b^2-2ab\sin(\alpha-\beta).$$

12. Here $$x+y=3-(1-2\sin^2 2\theta)=2+2\sin^2 2\theta,$$

and $$x-y=4\sin 2\theta;$$

$$\therefore\ x=1+2\sin 2\theta+\sin^2 2\theta=(1+\sin 2\theta)^2,$$

or $$x^{\frac{1}{2}}=1+\sin 2\theta.$$

Similarly, $$y^{\frac{1}{2}}=1-\sin 2\theta.$$

$$\therefore\ x^{\frac{1}{2}}+y^{\frac{1}{2}}=2.$$

13. By addition and subtraction, we have

$$x+y=(\sin\theta+\cos\theta)(1+\sin 2\theta)$$
$$=(\sin\theta+\cos\theta)^3,$$

and
$$x-y=(\sin\theta-\cos\theta)(1-\sin 2\theta)$$
$$=(\sin\theta-\cos\theta)^3;$$

$$\therefore\ (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2$$
$$=2.$$

14. Here
$$a^2=1+\sin 2\theta,$$
$$\therefore\ a^2-1+\cos 2\theta=b.$$
$$\therefore\ \sin 2\theta=a^2-1, \text{ and } \cos 2\theta=-(a^2-b-1);$$
$$\therefore\ (a^2-1)^2+(a^2-b-1)^2=1.$$

15. Here
$$a=(4\cos^3\theta-3\cos\theta)+(3\sin\theta-4\sin^3\theta)$$
$$=(\cos\theta-\sin\theta)\{4(\cos^2\theta+\cos\theta\sin\theta+\sin^2\theta)-3\}$$
$$=b(1+2\sin 2\theta).$$

But
$$1-\sin 2\theta=b^2;$$
$$\therefore\ a=b(1+2-2b^2)=3b-2b^3.$$

16. By squaring the first equation, and multiplying by 2, we have

$$a^2(1+\cos 2\theta)+b^2(1-\cos 2\theta)-2ab\sin 2\theta=2c^2;$$
$$\therefore\ (a^2-b^2)\cos 2\theta-2ab\sin 2\theta=2c^2-a^2-b^2.$$

From the second equation,

$$(a^2-b^2)\sin 2\theta+2ab\cos 2\theta=2c^2.$$

By squaring and adding, we obtain

$$(a^2-b^2)^2+4a^2b^2=4c^4-4c^2(a^2+b^2)+(a^2+b^2)^2+4c^4;$$
$$\therefore\ 0=8c^4-4c^2(a^2+b^2);$$
$$\therefore\ a^2+b^2=2c^2.$$

17. By squaring and adding, we have

$$x^2+y^2=a^2+b^2+2ab(\cos\theta\cos 2\theta+\sin\theta\sin 2\theta)$$
$$=a^2+b^2+2ab\cos\theta.$$

Again,
$$x+b=a\cos\theta+2b\cos^2\theta=\cos\theta(a+2b\cos\theta),$$

and
$$y=\sin\theta(a+2b\cos\theta);$$
$$\therefore\ (x+b)^2+y^2=(a+2b\cos\theta)^2$$
$$=\frac{1}{a^2}(x^2+y^2-b^2)^2.$$

18. Componendo and dividendo, we have

$$\frac{a}{b}=\frac{\tan(\theta+\alpha)+\tan(\theta-\alpha)}{\tan(\theta+\alpha)-\tan(\theta-\alpha)}=\frac{\sin 2\theta}{\sin 2\alpha};$$

$$\therefore\ b\sin 2\theta=a\sin 2\alpha.$$

Also $$b\cos 2\theta=c-a\cos 2\alpha;$$

$$\therefore\ b^2=c^2-2ac\cos 2\alpha+a^2.$$

19. By squaring and adding, we have

$$\begin{aligned}x^2+y^2&=a^2\{2-2(\cos 3\theta\cos\theta+\sin 3\theta\sin\theta)\}\\&=a^2(2-2\cos 2\theta)\\&=4a^2\sin^2\theta.\end{aligned}$$

And $$2a^2-x^2-y^2=2a^2\cos 2\theta.$$

But $$x=2a\cos 2\theta\sin\theta;$$

$$\begin{aligned}\therefore\ 4a^4x^2&=(2a^2\cos 2\theta)^2(4a^2\sin^2\theta)\\&=(2a^2-x^2-y^2)^2(x^2+y^2).\end{aligned}$$

20. Solving the given equations for x and y, we have

$$x=a(\cos\theta\cos 2\theta+2\sin\theta\sin 2\theta);$$

$$\begin{aligned}\therefore\ 2x&=a\{(\cos 3\theta+\cos\theta)+2(\cos\theta-\cos 3\theta)\}\\&=a(3\cos\theta-\cos 3\theta)=a(6\cos\theta-4\cos^3\theta).\end{aligned}$$

And $$y=a(2\sin 2\theta\cos\theta-\cos 2\theta\sin\theta);$$

$$\begin{aligned}\therefore\ 2y&=a\{2(\sin 3\theta+\sin\theta)-(\sin 3\theta-\sin\theta)\}\\&=a(3\sin\theta+\sin 3\theta)=a(6\sin\theta-4\sin^3\theta).\end{aligned}$$

$$\begin{aligned}\therefore\ 2(x+y)&=a(\cos\theta+\sin\theta)\{6-4(\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta)\}\\&=2a(\cos\theta+\sin\theta)(1+\sin 2\theta);\end{aligned}$$

$$\therefore\ x+y=a(\cos\theta+\sin\theta)^3.$$

Similarly, $$x-y=a(\cos\theta-\sin\theta)^3;$$

$$\therefore\ (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

21. From the first equation,

$$(x\sin\theta-y\cos\theta)^2=(x^2+y^2)(\sin^2\theta+\cos^2\theta);$$

whence $$x\cos\theta+y\sin\theta=0;$$

$$\therefore\ \frac{\cos\theta}{y}=\frac{\sin\theta}{-x}=\frac{1}{\sqrt{x^2+y^2}}.$$

By substituting in the second equation, we have

$$\frac{1}{x^2+y^2}\left(\frac{y^2}{a^2}+\frac{x^2}{b^2}\right)=\frac{1}{x^2+y^2};$$

or $$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

22. From the first equation, we have

$$bx\cos\theta + ay\sin\theta = ab\sqrt{\sin^2\theta + \cos^2\theta};$$

$$\therefore\ b^2(x^2-a^2)\cos^2\theta + 2abxy\cos\theta\sin\theta + a^2(y^2-b^2)\sin^2\theta = 0.$$

From the second equation, we have

$$(y^2-b^2)\cos^2\theta - 2xy\cos\theta\sin\theta + (x^2-a^2)\sin^2\theta = 0.$$

Multiplying this equation by ab and adding to the preceding equation,

$$\{b^2(x^2-a^2) + ab(y^2-b^2)\}\cos^2\theta + \{a^2(y^2-b^2) + ab(x^2-a^2)\}\sin^2\theta = 0;$$

$$\therefore\ \{bx^2 + ay^2 - ab(a+b)\}(b\cos^2\theta + a\sin^2\theta) = 0.$$

$$\therefore\ bx^2 + ay^2 - ab(a+b) = 0,$$

or

$$b\cos^2\theta + a\sin^2\theta = 0.$$

From the last result,

$$\frac{\cos^2\theta}{a} = \frac{\sin^2\theta}{-b};$$

but

$$\{(y^2-b^2)\cos^2\theta + (x^2-a^2)\sin^2\theta\}^2 = 4x^2y^2\cos^2\theta\sin^2\theta;$$

$$\therefore\ \{a(y^2-b^2) - b(x^2-a^2)\}^2 = -4abx^2y^2.$$

23. We have

$$4\cos(\alpha - 3\theta) = m(3\cos\theta + \cos 3\theta),$$

and

$$4\sin(\alpha - 3\theta) = m(3\sin\theta - \sin 3\theta);$$

whence by squaring and adding, we have

$$16 = m^2(10 + 6\cos 4\theta).$$

Again, by multiplying the first equation by $\cos 3\theta$ and the second by $\sin 3\theta$ and subtracting, we obtain

$$4\cos\alpha = m(3\cos 4\theta + 1);$$

$$\therefore\ 16 = 10m^2 + 2m(4\cos\alpha - m),$$

or

$$2 = m^2 + m\cos\alpha.$$

24. We have

$$\frac{x}{y} = \frac{\tan\theta + \tan\phi}{\cot\theta + \cot\phi} = \tan\theta\tan\phi,$$

But

$$\tan\alpha = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi};$$

$$\therefore\ \tan\alpha = \frac{xy}{y-x}.$$

25. We have

$$a^2 + b^2 = 2 + 2\cos(\theta - \phi);$$

$$\therefore\ a^2 + b^2 = 2 + 2\cos\alpha.$$

26. We have

$$a\sin^2\theta + b\cos^2\theta = 1 = \sin^2\theta + \cos^2\theta;$$

$$\therefore\ (a-1)\tan^2\theta = 1 - b.$$

Again, $$a\cos^2\phi + b\sin^2\phi = 1 = \cos^2\phi + \sin^2\phi;$$
$$\therefore\ (b-1)\tan^2\phi = 1-a.$$

But $$a^2\tan^2\theta = b^2\tan^2\phi;$$
$$\therefore\ \frac{a^2(1-b)}{a-1} = \frac{b^2(1-a)}{b-1};$$
$$\therefore\ a(b-1) = \pm b(a-1).$$

Rejecting the upper sign, we have $a+b=2ab$.

27. From the first two equations, we have

$$\frac{\frac{x}{a}}{\cos\frac{\theta+\phi}{2}} = \frac{\frac{y}{b}}{\sin\frac{\theta+\phi}{2}} = \frac{1}{\cos\frac{\theta-\phi}{2}};$$
$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\frac{\theta-\phi}{2} = \sec^2\frac{\alpha}{2}.$$

28. We have $$\frac{a}{b} = \frac{\tan\theta + \tan\phi}{\cot\theta + \cot\phi} = \tan\theta\tan\phi.$$

But $$\tan\alpha = \tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi};$$
$$\therefore\ (a+b)\tan\alpha = b(\tan\theta - \tan\phi);$$
$$\therefore\ (a+b)^2\tan^2\alpha = b^2\{(\tan\theta+\tan\phi)^2 - 4\tan\theta\tan\phi\}$$
$$= b^2a^2 - 4ab.$$

29. We have $$\frac{a\cos^2\theta + b\sin^2\theta}{a\sin^2\theta + b\cos^2\theta} = \frac{m\cos^2\phi}{n\sin^2\phi};$$
$$\therefore\ \frac{a+b\tan^2\theta}{a\tan^2\theta+b} = \frac{m}{n\tan^2\phi} = \frac{1}{\tan^2\theta};$$
$$\therefore\ b\tan^4\theta = b, \text{ or } \tan^2\theta = \pm 1.$$
$$\therefore\ n\tan^2\phi = \pm m.$$

By adding together the first two equations, we obtain
$$a+b = m\cos^2\phi + n\sin^2\phi.$$

If $n\tan^2\phi = m$, then $$\frac{\cos^2\phi}{n} = \frac{\sin^2\phi}{m} = \frac{1}{m+n};$$
$$\therefore\ a+b = \frac{2mn}{m+n}.$$

If $n\tan^2\phi = -m$, we obtain $a+b=0$.

30. From the second and third equations, we have by addition

$$2\cos\phi\,(x\cos\theta+y\sin\theta)=6a;$$

$$\therefore\ 2a\sqrt{3}\cos\phi=3a, \text{ whence } \cos\phi=\frac{\sqrt{3}}{2}.$$

From the second and third equations, we have by subtraction

$$2\sin\phi\,(-x\sin\theta+y\cos\theta)=2a;$$

$$\therefore\ -x\sin\theta+y\cos\theta=\pm 2a.$$

But

$$x\cos\theta+y\sin\theta=2a\sqrt{3};$$

$$\therefore\ x^2+y^2=16a^2.$$

31. We have $c\sin\theta=a\,(\sin\theta\cos\phi+\cos\theta\sin\phi)$;

$$\therefore\ (c-a\cos\phi)\sin\theta=a\sin\phi\cos\theta=b\sin\theta\,.\,\cos\theta;$$

$$\therefore\ c-a\cos\phi=b\cos\theta=b\,(2m+\cos\phi);$$

$$\therefore\ \cos\phi=\frac{c-2bm}{a+b}.$$

And

$$\cos\theta=2m+\cos\phi=\frac{c+2am}{a+b}.$$

But

$$a\sin\phi=b\sin\theta;$$

$$\therefore\ a^2-a^2\cos^2\phi=b^2-b^2\cos^2\theta;$$

$$\therefore\ a^2(a+b)^2-a^2(c-2bm)^2=b^2(a+b)^2-b^2(c+2am)^2;$$

$$\therefore\ (a^2-b^2)(a+b)^2=c^2(a^2-b^2)-4abcm(a+b);$$

$$\therefore\ 4abcm=(a-b)\{c^2-(a+b)^2\}.$$

EXAMPLES. XXV. c. PAGE 334.

1. By putting $x=y\cos\theta$, the given equation becomes

$$\cos^3\theta-\frac{3}{y^2}\cos\theta-\frac{1}{y^3}=0.$$

But

$$\cos^3\theta-\frac{3}{4}\cos\theta-\frac{\cos 3\theta}{4}=0;$$

$$\therefore\ \frac{3}{y^2}=\frac{3}{4};\ \text{whence } y=2.$$

Also

$$\frac{\cos 3\theta}{4}=\frac{1}{y^3}=\frac{1}{8};\ \text{whence } \cos 3\theta=\frac{1}{2};$$

$$\therefore\ 3\theta=n\,.\,360°\pm 60°;$$

$$\therefore\ \theta=20°,\ 100°,\ \text{or } 140°.$$

But $x=y\cos\theta=2\cos\theta$; and therefore the roots are

$$2\cos 20°,\quad -2\cos 40°.\quad -2\cos 80°.$$

2. By putting $x=y\sin\theta$, the given equation becomes

$$\sin^3\theta-\frac{3}{y^2}\sin\theta+\frac{1}{y^3}=0.$$

But $$\sin^3\theta-\frac{3}{4}\sin\theta+\frac{\sin 3\theta}{4}=0;$$

$$\therefore\ y^2=4;\ \text{whence } y=2.$$

Also $$\frac{\sin 3\theta}{4}=\frac{1}{y^3}=\frac{1}{8};\ \text{whence } \sin 3\theta=\frac{1}{2}.$$

Hence as in Art. 328, the roots of the equation are

$$2\sin 10°,\quad 2\sin 50°,\quad -2\sin 70°.$$

3. Put $x=y\cos\theta$, then $\cos^3\theta-\frac{3}{y^2}\cos\theta-\frac{\sqrt{3}}{y^3}=0.$

But $$\cos^3\theta-\frac{3}{4}\cos\theta-\frac{\cos 3\theta}{4}=0;$$

$$\therefore\ y^2=4;\ \text{whence } y=2.$$

Also $$\frac{\cos 3\theta}{4}=\frac{\sqrt{3}}{y^3}=\frac{\sqrt{3}}{8};\ \text{whence } \cos 3\theta=\frac{\sqrt{3}}{2};$$

$$\therefore\ 3\theta=n\,.\,360°\pm 30°;\quad \therefore\ \theta=10°,\ 110°,\ \text{or } 130°.$$

But $x=y\cos\theta=2\cos\theta$, and therefore the roots are

$$2\cos 10°,\quad -2\cos 50°,\quad -2\cos 70°.$$

4. Put $x=y\sin\theta$; then $\sin^3\theta-\frac{3}{4y^2}\sin\theta+\frac{\sqrt{2}}{8y^3}=0.$

But $$\sin^3\theta-\frac{3}{4}\sin\theta+\frac{\sin 3\theta}{4}=0;$$

$$\therefore\ 4y^2=4;\ \text{whence } y=1.$$

Also since $\frac{\sin 3\theta}{4}=\frac{\sqrt{2}}{8y^3}=\frac{\sqrt{2}}{8}$; whence $\sin 3\theta=\frac{1}{\sqrt{2}}$;

$$\therefore\ 3\theta=n\,.\,180°+(-1)^n 45°;$$

$$\therefore\ \theta=15°,\ 45°,\ 135°,\ 165°,\ 255°,\ \ldots.$$

$$\therefore\ \sin\theta=\sin 15°,\ \sin 45°,\ \sin 255°.$$

But $x=y\sin\theta=\sin\theta$, and therefore the roots are

$$\sin 15°,\quad \sin 45°,\quad -\sin 75°.$$

5. Put $x=y\sin\theta$; then $\sin^3\theta-\frac{3}{4a^2y^2}\sin\theta+\frac{\sin 3\theta}{4a^3y^3}=0.$

But $$\sin^3\theta-\frac{3}{4}\sin\theta+\frac{\sin 3\theta}{4}=0;$$

$$\therefore\ 4a^2y^2=4;\ \text{whence } y=\frac{1}{a}.$$

Also $$\frac{\sin 3\theta}{4} = \frac{\sin 3A}{4a^3y^3} = \frac{\sin 3A}{4};$$

$$\therefore\ 3\theta = n \,.\, 180° + (-1)^n 3A;$$

$$\therefore\ \theta = A,\quad 60° - A,\quad 120° + A,\quad 180° - A,\quad 240° + A, \ldots$$

$$\therefore\ \sin\theta = \sin A,\quad \sin(60° - A),\quad \sin(240° + A).$$

But $x = y\sin\theta = \frac{1}{a}\sin\theta$, and therefore the roots are

$$\frac{1}{a}\sin A,\quad \frac{1}{a}\sin(60° - A),\quad -\frac{1}{a}\sin(60° + A).$$

6. Put $x = y\cos\theta$; then $\cos^3\theta - \frac{3a^2}{y^2}\cos\theta - \frac{2a^3\cos 3A}{y^3} = 0.$

But $$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{\cos 3\theta}{4} = 0;$$

$$\therefore\ \frac{3a^2}{y^2} = \frac{3}{4};\ \text{whence } y = 2a.$$

Also $$\frac{\cos 3\theta}{4} = \frac{2a^3\cos 3A}{y^3} = \frac{\cos 3A}{4};$$

$$\therefore\ 3\theta = n \,.\, 360° \pm 3A;$$

$$\therefore\ \theta = A,\ 120° \pm A.$$

But $x = y\cos\theta = 2a\cos\theta$, and therefore the roots are

$$2a\cos A,\quad 2a\cos(120° \pm A).$$

7. (1) From the theory of quadratic equations, we have

$$\sin\alpha + \sin\beta = -\frac{b}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

By supposition, $$\sin\alpha + 2\sin\beta = 1,$$

$$\therefore\ \sin\beta = 1 + \frac{b}{a}.$$

But $$a\sin^2\beta + b\sin\beta + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

$$\therefore\ (a+b)^2 + b(a+b) + ac = 0.$$

(2) Substituting from the equation $c\sin\alpha = a\sin\beta$ in (1), we have

$$a\sin\beta + c\sin\beta = -\frac{bc}{a};$$

$$\therefore\ a(a+c)\sin\beta = -bc.$$

But $$\sin\alpha\sin\beta = \frac{c}{a};$$

$$\therefore\ a^2\sin^2\beta = c^2;\ \text{whence } a + c = \pm b.$$

8. We have $\tan\alpha+\tan\beta=\frac{b}{a}$.

Also, by hypothesis, $a\tan\alpha+b\tan\beta=2b$;

whence $(b-a)\tan\beta=b$.

But $a\tan^2\beta-b\tan\beta+c=0$;

$\therefore\ ab^2-b^2(b-a)+c(b-a)^2=0.$

9. We have $\tan\alpha+\tan\beta+\tan\gamma=0,$

and $\tan\alpha\tan\beta+\tan\alpha\tan\gamma+\tan\beta\tan\gamma=\frac{2a-x}{a}$;

$\therefore\ \tan\alpha\tan\beta-\tan^2\gamma=\frac{2a-x}{a}$.

Now $\tan^2\alpha+\tan^2\beta=(\tan\alpha+\tan\beta)^2-2\tan\alpha\tan\beta$

$=\tan^2\gamma-2\tan^2\gamma-\frac{2(2a-x)}{a}$;

$\therefore\ a(\tan^2\alpha+\tan^2\beta)=-a\tan^2\gamma-4a+2x.$

But, by hypothesis, $a(\tan^2\alpha+\tan^2\beta)=2x-5a$;

$\therefore\ 2x-5a=-a\tan^2\gamma-4a+2x$;

$\therefore\ \tan^2\gamma=1.$

But $a\tan^3\gamma+(2a-x)\tan\gamma+y=0$;

$\therefore\ a\tan\gamma+(2a-x)\tan\gamma+y=0$;

$\therefore\ (3a-x)\tan\gamma+y=0$;

$\therefore\ 3a-x=\pm y.$

10. We have $\cos\alpha\cos\beta+\cos\alpha\cos\gamma+\cos\beta\cos\gamma=b.$

Also, by supposition, $\cos\alpha\cos\beta+\cos\alpha\cos\gamma=2b$;

$\therefore\ \cos\beta\cos\gamma=-b.$

Again, $\cos\alpha\cos\beta\cos\gamma=-c$;

$\therefore\ b\cos\alpha=c.$

Now $\cos^3\alpha+a\cos^2\alpha+b\cos\alpha+c=0$;

$\therefore\ c^3+abc^2+b^3c+b^3c=0.$

11. As on page 329 we may shew that

$$\cos\alpha,\quad \cos\left(\frac{2\pi}{3}+\alpha\right),\quad \cos\left(\frac{2\pi}{3}-\alpha\right)$$

are the roots of the cubic

$$\cos^3\theta-\frac{3}{4}\cos\theta-\frac{\cos 3\alpha}{4}=0.$$

Write $\cos\theta=\frac{1}{\sec\theta}$; then $\frac{\cos 3\alpha}{4}\sec^3\theta+\frac{3}{4}\sec^2\theta-1=0$;

$$\therefore\ \sec\alpha+\sec\left(\frac{2\pi}{3}+\alpha\right)+\sec\left(\frac{2\pi}{3}-\alpha\right)=-\frac{3}{4}\div\frac{\cos 3\alpha}{4}.$$

12. The values of $\sin\theta$ found from the equation $\sin 3\theta=\sin 3\alpha$ are

$$\sin\alpha,\quad \sin\left(\frac{2\pi}{3}+\alpha\right),\quad \sin\left(\frac{4\pi}{3}+\alpha\right);$$

that is, these three quantities are the roots of the cubic equation

$$\sin^3\theta-\frac{3}{4}\sin\theta+\frac{\sin 3\alpha}{4}=0 \qquad \text{.....(1).}$$

$\therefore\ S_1=$ Sum of the roots $=0$;

and $S_2=$ Sum of products of the roots taken two at a time $=-\frac{3}{4}$;

$$\therefore\ \sin^2\alpha+\sin^2\left(\frac{2\pi}{3}+\alpha\right)+\sin^2\left(\frac{4\pi}{3}+\alpha\right)=S_1^2-2S_2$$
$$=0-2\left(-\frac{3}{4}\right)=\frac{3}{2}.$$

13. In equation (1) of Example 12, put $\sin\theta=\frac{1}{\operatorname{cosec}\theta}$; then

$$\frac{\sin 3\alpha}{4}\operatorname{cosec}^3\theta-\frac{3}{4}\operatorname{cosec}^2\theta+1=0.$$

$$\therefore\ \operatorname{cosec}\alpha+\operatorname{cosec}\left(\frac{2\pi}{3}+\alpha\right)+\operatorname{cosec}\left(\frac{4\pi}{3}+\alpha\right)=\frac{3}{4}\div\frac{\sin 3\alpha}{4}.$$

14. On p. 330 we have shewn that $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{2\pi}{5}$ are the roots of the quadratic equation $16x^2-20x+5=0$.

Put $x=\frac{1}{y}$; then $5y^2-20y+16=0$, is an equation whose roots are

$$\operatorname{cosec}^2\frac{\pi}{5}\text{ and }\operatorname{cosec}^2\frac{2\pi}{5};$$

$$\therefore\ \operatorname{cosec}^2\frac{\pi}{5}+\operatorname{cosec}^2\frac{2\pi}{5}=\frac{20}{5}=4.$$

15. If $$\cos 3\theta = \cos 2\theta,$$

then $$4\cos^3\theta - 2\cos^2\theta - 3\cos\theta + 1 = 0,$$

or $$(\cos\theta - 1)(4\cos^2\theta + 2\cos\theta - 1) = 0.$$

But the roots of $\cos 3\theta = \cos 2\theta$ considered as a cubic in $\cos\theta$ are

$$1, \quad \cos\frac{2\pi}{5}, \quad \cos\frac{4\pi}{5}. \qquad \text{[Art. 331.]}$$

$$\therefore \cos\frac{2\pi}{5} \text{ and } \cos\frac{4\pi}{5} \text{ are the roots of } 4\cos^2\theta + 2\cos\theta - 1 = 0;$$

$$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{2}{4} = -\frac{1}{2}.$$

and $$\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4}.$$

16. (1) Let $7\theta = n\pi$, where n is any *odd* integer;

then $$4\theta = n\pi - 3\theta, \text{ and } \cos 4\theta = -\cos 3\theta.$$

$$\therefore 8\cos^4\theta - 8\cos^2\theta + 1 = -4\cos^3\theta + 3\cos\theta;$$

$$\therefore (\cos\theta + 1)(8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The roots of the equation $\cos 4\theta = -\cos 3\theta$ considered as a biquadratic in $\cos\theta$ are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$, $\cos\frac{7\pi}{7}$, the last of which corresponds to the factor $\cos\theta + 1$ in equation (1). Hence the equation whose roots are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, $\cos\frac{5\pi}{7}$ is

$$8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0.$$

(2) Let y denote any one of the quantities

$$\sin^2\frac{\pi}{14}, \quad \sin^2\frac{3\pi}{14}, \quad \sin^2\frac{5\pi}{14}.$$

then $2y = 1 - x$, where x denotes one of the quantities

$$\cos\frac{\pi}{7}, \quad \cos\frac{3\pi}{7}, \quad \cos\frac{5\pi}{7}.$$

But we have seen in the first part of this question that these quantities are the roots of the cubic

$$8x^3 - 4x^2 - 4x + 1 = 0.$$

Substituting $x = 1 - 2y$, we have

$$8(1-2y)^3 - 4(1-2y)^2 - 4(1-2y) + 1 = 0,$$

or $$64y^3 - 80y^2 + 24y - 1 = 0.$$

17. Let y denote one of the quantities $\sin^2\frac{\pi}{7}$, $\sin^2\frac{2\pi}{7}$, $\sin^2\frac{3\pi}{7}$, then $2y=1-x$; where x denotes one of the quantities

$$\cos\frac{2\pi}{7},\ \cos\frac{4\pi}{7},\ \cos\frac{6\pi}{7}.$$

But in Art. 331, we have shewn that these quantities are the roots of the cubic

$$8x^3+4x^2-4x-1=0.$$

Substituting $x=1-2y$, we have

$$8(1-2y)^3+4(1-2y)^2-4(1-2y)-1=0,$$

or

$$64y^3-112y^2+56y-7=0,$$

the roots of which are $\sin^2\frac{\pi}{7}$, $\sin^2\frac{2\pi}{7}$, $\sin^2\frac{3\pi}{7}$.

$$\therefore\ \sin^4\frac{\pi}{7}+\sin^4\frac{2\pi}{7}+\sin^4\frac{3\pi}{7}=\left(\frac{112}{64}\right)^2-2\left(\frac{56}{64}\right)=\frac{21}{16}.$$

Put $y=\frac{1}{z}$; then $7z^3-56z^2+112z-64=0$ is an equation whose roots are

$$\operatorname{cosec}^2\frac{\pi}{7},\ \operatorname{cosec}^2\frac{2\pi}{7},\ \operatorname{cosec}^2\frac{3\pi}{7}.$$

$$\therefore\ \operatorname{cosec}^4\frac{\pi}{7}+\operatorname{cosec}^4\frac{2\pi}{7}+\operatorname{cosec}^4\frac{3\pi}{7}=\left(\frac{56}{7}\right)^2-2\left(\frac{112}{7}\right)=32.$$

18. (1) As in Art. 331 the required equation is $\cos 5\theta=\cos 4\theta$.

Expressing $\cos 5\theta$ and $\cos 4\theta$ in terms of $\cos\theta$, we have

$$16\cos^5\theta-20\cos^3\theta+5\cos\theta=8\cos^4\theta-8\cos^2\theta+1.\quad\text{[See Art. 332.]}$$

By transposition and removal of the factor $\cos\theta-1$ we obtain

$$16\cos^4\theta+8\cos^3\theta-12\cos^2\theta-4\cos\theta+1=0,$$

which is the equation required.

(2) Put $\cos\theta=x$, then the above equation becomes

$$16x^4+8x^3-12x^2-4x+1=0.$$

Now

$$\cos\frac{\pi}{9}=-\cos\frac{8\pi}{9},\ \cos\frac{3\pi}{9}=-\cos\frac{6\pi}{9},\ \ldots;$$

nce by writing $x=-y$, we see that

$$\cos\frac{\pi}{9},\ \cos\frac{3\pi}{9},\ \cos\frac{5\pi}{9},\ \cos\frac{7\pi}{9}$$

are the roots of the equation $16y^4-8y^3-12y^2+4y+1=0$.

The same result may be arrived at by putting $9\theta=n\pi$, where n is any *odd ger.* For $5\theta=n\pi-4\theta$, so that $\cos 5\theta=-\cos 4\theta$. [See Example 16 (1).]

19. Let y denote one of the quantities

$$\cos^2\frac{\pi}{9},\ \cos^2\frac{2\pi}{9},\ \cos^2\frac{3\pi}{9},\ \cos^2\frac{4\pi}{9} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

then $2y=1+x$, where x denotes one of the quantities

$$\cos\frac{2\pi}{9},\ \cos\frac{4\pi}{9},\ \cos\frac{6\pi}{9},\ \cos\frac{8\pi}{9}.$$

But these quantities are by the last example the roots of the equation

$$16x^4+8x^3-12x^2-4x+1=0;$$

hence the given quantities are the roots of

$$16(2y-1)^4+8(2y-1)^3-12(2y-1)^2-4(2y-1)+1=0,$$

or
$$256y^4-448y^3+240y^2-40y+1=0.$$

$$\therefore \cos^4\frac{\pi}{9}+\cos^4\frac{2\pi}{9}+\cos^4\frac{3\pi}{9}+\cos^4\frac{4\pi}{9}=\left(\frac{448}{256}\right)^2-2\left(\frac{240}{256}\right)$$

$$=\frac{49}{16}-\frac{30}{16}=\frac{19}{16}.$$

If we put $y=\frac{1}{z}$, we obtain

$$z^4-40z^3+240z^2-448z+256=0,$$

the roots of which are

$$\sec^2\frac{\pi}{9},\ \sec^2\frac{2\pi}{9},\ \sec^2\frac{3\pi}{9},\ \sec^2\frac{4\pi}{9}.$$

$$\therefore \sec^4\frac{\pi}{9}+\sec^4\frac{2\pi}{9}+\sec^4\frac{3\pi}{9}+\sec^4\frac{4\pi}{9}=(40)^2-2\times240=1120.$$

20. As on p. 332, we may shew that the given quantities are the roots of the equation $\tan 5\theta=-\tan 4\theta$, considered as an equation in $\tan\theta$.

Put $\tan\theta=t$, then $\tan 5\theta=\tan(3\theta+2\theta)$;

$$\therefore \tan 5\theta=\frac{\dfrac{3t-t^3}{1-3t^2}+\dfrac{2t}{1-t^2}}{1-\dfrac{2t(3t-t^3)}{(1-3t^2)(1-t^2)}}=\frac{5t-10t^3+t^5}{1-10t^2+5t^4}.$$

Hence the equation $\tan 5\theta=-\tan 4\theta$ becomes

$$\frac{t^5-10t^3+5t}{5t^4-10t^2+1}-\frac{4t^3-4t}{t^4-6t^2+1}=0;$$

$$\therefore (t^8-16t^6+66t^4-40t^2+5)-(20t^6-60t^4+4t^2-4)=0;$$

$$\therefore t^8-36t^6+126t^4-84t^2+9=0,$$

which is a biquadratic in t^2 having for roots

$$\tan^2\frac{\pi}{9},\ \tan^2\frac{2\pi}{9},\ \tan^2\frac{3\pi}{9},\ \tan^2\frac{4\pi}{9}.$$

Put $t^2=\frac{1}{x}$, then

$$9x^4-84x^3+126x^2-36+1=0,$$

an equation whose roots are

$$\cot^2\frac{\pi}{9},\ \cot^2\frac{2\pi}{9},\ \cot^2\frac{3\pi}{9},\ \cot^2\frac{4\pi}{9}.$$

But $\cot^2\frac{3\pi}{9}=\cot^2\frac{\pi}{3}=\frac{1}{3}$.

$$\therefore\ \cot^2\frac{\pi}{9}+\cot^2\frac{2\pi}{9}+\cot^2\frac{4\pi}{9}+\frac{1}{3}=\frac{84}{9}=9\tfrac{1}{3}.$$

$$\therefore\ \cot^2\frac{\pi}{9}+\cot^2\frac{2\pi}{9}+\cot^2\frac{4\pi}{9}=9.$$

21. (1) As in Ex. 1, p. 332, we can shew that $t^6-21t^4+35t^2-7=0$ is an equation whose roots are

$$\tan^2\frac{\pi}{7},\ \tan^2\frac{2\pi}{7},\ \tan^2\frac{3\pi}{7};$$

$\therefore$ writing $\frac{1}{c}$ for t, we obtain

$$7c^6-35c^4+21c^2-1=0,$$

which is a cubic in c^2 whose roots are

$$\cot^2\frac{\pi}{7},\ \cot^2\frac{2\pi}{7},\ \cot^2\frac{3\pi}{7}.$$

$$\therefore\ \cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}=5;$$

$$\therefore\ \operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}=8.$$

(2) Here $\cos\frac{\pi}{11}=-\cos\frac{10\pi}{11}$, $\cos\frac{3\pi}{11}=-\cos\frac{8\pi}{11}$, $\cos\frac{5\pi}{11}=-\cos\frac{6\pi}{11}$.

$$\therefore\ \cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}$$

$$=-\cos\frac{2\pi}{11}\cos\frac{4\pi}{11}\cos\frac{6\pi}{11}\cos\frac{8\pi}{11}\cos\frac{10\pi}{11}.$$

$\cos\frac{2\pi}{11}$, $\cos\frac{4\pi}{11}$, ... $\cos\frac{10\pi}{11}$ are the roots of the equation

$$\cos 6\theta=\cos 5\theta.$$ [See Art. 331.]

Using the expressions for $\cos 6\theta$ and $\cos 5\theta$ given in Art. 332 and putting x for $\cos\theta$, we have

$$32x^6 - 48x^4 + 18x^2 - 1 = 16x^5 - 20x^3 + 5x,$$

or
$$32x^6 - 16x^5 - 48x^4 + 20x^3 + 18x^2 - 5x - 1 = 0.$$

Removing the factor $x-1$, which corresponds to $\cos\theta = 1$, we have

$$32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 = 0.$$

The product of the roots is $-\frac{1}{32}$, and therefore the value of the required expression is $\frac{1}{32}$.

MISCELLANEOUS EXAMPLES. I. PAGE 336.

1. By transposition, we have

$$a\left(\tan\alpha - \tan\frac{\alpha+\beta}{2}\right) = b\left(\tan\frac{\alpha+\beta}{2} - \tan\beta\right);$$

$$\therefore \frac{a\sin\left(\alpha - \frac{\alpha+\beta}{2}\right)}{\cos\alpha\cos\frac{\alpha+\beta}{2}} = \frac{b\sin\left(\frac{\alpha+\beta}{2} - \beta\right)}{\cos\beta\cos\frac{\alpha+\beta}{2}};$$

$$\therefore \frac{a}{\cos\alpha} = \frac{b}{\cos\beta}.$$

2. We have $$\frac{a+b}{a}\sin^4\alpha + \frac{a+b}{b}\cos^4\alpha = 1;$$

$$\therefore \left(1+\frac{b}{a}\right)\sin^4\alpha + \left(1+\frac{a}{b}\right)\cos^4\alpha = \sin^4\alpha + 2\sin^2\alpha\cos^2\alpha + \cos^4\alpha;$$

$$\therefore \frac{b}{a}\sin^4\alpha - 2\sin^2\alpha\cos^2\alpha + \frac{a}{b}\cos^4\alpha = 0;$$

$$\therefore \frac{\sin^4\alpha}{a^2} - \frac{2\sin^2\alpha\cos^2\alpha}{ab} + \frac{\cos^4\alpha}{b^2} = 0;$$

$$\therefore \frac{\sin^2\alpha}{a} = \frac{\cos^2\alpha}{b} = \frac{1}{a+b}.$$

$$\therefore \frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3} = \frac{a}{(a+b)^4} + \frac{b}{(a+b)^4} = \frac{1}{(a+b)^3}.$$

3. The left side $= \tan^{-1} \dfrac{2\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$

$$= \tan^{-1} \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{\cos^2\frac{\alpha}{2}\cos^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)-\sin^2\frac{\alpha}{2}\sin^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}$$

$$= \tan^{-1} \frac{\frac{1}{2}\sin\alpha\sin\left(\frac{\pi}{2}-\beta\right)}{\cos\left\{\frac{\alpha}{2}+\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}\cos\left\{\frac{\alpha}{2}-\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}}$$

[XI. a. Ex. 12.]

$$= \tan^{-1} \frac{\sin\alpha\cos\beta}{\cos\alpha+\cos\left(\frac{\pi}{2}-\beta\right)}.$$

4. By putting $n=1$, we have

$$\operatorname{cosec}^2\alpha\sin^4\theta+\sec^2\alpha\cos^4\theta=1;$$

$$\therefore (1+\cot^2\alpha)\sin^4\theta+(1+\tan^2\alpha)\cos^4\theta=\sin^4\theta+2\sin^2\theta\cos^2\theta+\cos^4\theta;$$

$$\therefore \cot^2\alpha\sin^4\theta-2\sin^2\theta\cos^2\theta+\tan^2\alpha\cos^4\theta=0;$$

$$\therefore \cot\alpha\sin^2\theta-\tan\alpha\cos^2\theta=0;$$

$$\therefore \frac{\sin^2\theta}{\sin^2\alpha}=\frac{\cos^2\theta}{\cos^2\alpha}=1;$$

$$\therefore \frac{\sin^{2n}\theta}{\sin^{2n}\alpha}=\frac{\cos^{2n}\theta}{\cos^{2n}\alpha}=1;$$

$$\therefore \frac{\sin^{2n+2}\theta}{\sin^{2n}\alpha}+\frac{\cos^{2n+2}\theta}{\cos^{2n}\alpha}=\sin^2\theta+\cos^2\theta=1.$$

5. We have $(a\cos\theta+b\sin\theta)^2=c^2=c^2(\cos^2\theta+\sin^2\theta)$;

$$\therefore (a^2-c^2)\cos^2\theta+2ab\cos\theta\sin\theta+(b^2-c^2)\sin^2\theta=0.$$

Again, $a\cos^2\theta+b\sin^2\theta=c(\cos^2\theta+\sin^2\theta)$;

$$\therefore (a-c)\cos^2\theta+(b-c)\sin^2\theta=0.$$

Hence by cross multiplication,

$$\frac{\cos^2\theta}{2ab(b-c)}=\frac{\cos\theta\sin\theta}{(b^2-c^2)(a-c)-(a^2-c^2)(b-c)}=\frac{\sin^2\theta}{-2ab(a-c)};$$

$$\therefore \frac{\cos^2\theta}{2ab(b-c)}=\frac{\cos\theta\sin\theta}{(b-c)(a-c)(b-a)}=\frac{\sin^2\theta}{-2ab(a-c)};$$

$$\therefore -4a^2b^2(b-c)(a-c)=(b-c)^2(a-c)^2(b-a)^2,$$

or $$4a^2b^2+(b-c)(a-c)(a-b)^2=0.$$

6. (i) $4\Sigma \sin(\beta-\gamma)\cos(\alpha-\beta)\cos(\alpha-\gamma)$

$=2\Sigma \sin(\beta-\gamma)\{\cos(2\alpha-\beta-\gamma)+\cos(\beta-\gamma)\}$

$=2\Sigma \sin(\beta-\gamma)\cos\{2\alpha-(\beta+\gamma)\}+2\Sigma \sin(\beta-\gamma)\cos(\beta-\gamma)$

$=\Sigma\{\sin(2\alpha-2\gamma)+\sin(2\beta-2\alpha)\}+\Sigma \sin 2(\beta-\gamma)$

$=-2\Sigma \sin 2(\beta-\gamma)+\Sigma \sin 2(\beta-\gamma)$

$=-\Sigma \sin 2(\beta-\gamma)$

$=4\Pi \sin(\beta-\gamma)$. [Ex. 2, page 304.]

(ii) $4\Sigma \sin\alpha \sin(\beta-\gamma)\cos(\beta+\gamma-\alpha)$

$=2\Sigma \sin\alpha\{\sin(2\beta-\alpha)+\sin(\alpha-2\gamma)\}$

$=2\Sigma \sin\alpha \sin(2\beta-\alpha)+2\Sigma \sin\alpha \sin(\alpha-2\gamma)$

$=\Sigma\{\cos 2(\alpha-\beta)-\cos 2\beta\}+\Sigma\{\cos 2\gamma-\cos 2(\alpha-\gamma)\}$

$=\Sigma\{\cos 2(\alpha-\beta)-\cos 2(\gamma-\alpha)\}+\Sigma(\cos 2\gamma-\cos 2\beta)$

$=0$.

(iii) $4\Sigma \sin\alpha \sin(\beta-\gamma)\sin(\beta+\gamma-\alpha)$

$=2\Sigma \sin\alpha\{\cos(\alpha-2\gamma)-\cos(2\beta-\alpha)\}$

$=\Sigma\{\sin(2\alpha-2\gamma)+\sin 2\gamma\}-\Sigma\{\sin 2\beta+\sin(2\alpha-2\beta)\}$

$=-\Sigma \sin 2(\gamma-\alpha)+\Sigma(\sin 2\gamma-\sin 2\beta)-\Sigma \sin 2(\alpha-\beta)$

$=-2\Sigma \sin 2(\alpha-\beta)$

$=8\Pi \sin(\alpha-\beta)$. [Ex. 2, page 304.]

7. (1) Let x, y, z denote the lengths of PA, PB, PC respectively; and let the areas of the triangles PBC, PCA, PAB be denoted by $\delta_1, \delta_2, \delta_3$ respectively.

Then in the triangle PBC

$$\cot\omega=\frac{\cos\omega}{\sin\omega}=\frac{a^2+y^2-z^2}{2ay\sin\omega}=\frac{a^2+y^2-z^2}{4\delta_1};$$

$$\therefore \cot\omega=\frac{a^2+y^2-z^2}{4\delta_1}=\frac{b^2+z^2-x^2}{4\delta_2}=\frac{c^2+x^2-y^2}{4\delta_3}$$

$$=\frac{a^2+b^2+c^2}{4(\delta_1+\delta_2+\delta_3)}=\frac{a^2+b^2+c^2}{4\Delta}$$

$$=\cot A+\cot B+\cot C.$$ [XVIII. a., Ex. 34.]

(2) By squaring the result just obtained, we have

$$\cot^2\omega=\cot^2 A+\cot^2 B+\cot^2 C+2\Sigma \cot B\cot C$$

$$=\cot^2 A+\cot^2 B+\cot^2 C+2,$$

since $$\Sigma \cot B\cot C=1.$$

$$\therefore 1+\cot^2\omega=(1+\cot^2 A)+(1+\cot^2 B)+(1+\cot^2 C);$$

$$\therefore \operatorname{cosec}^2\omega=\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C.$$

8. On page 195 of the *Elementary Trigonometry*, suppose that $ABFD$ is a vertical plane running N. and S., and that CG is drawn in a S.E. direction. Also suppose that $AB=a$, and $AC=169a$;

then
$$BC^2=(169a)^2-a^2=28560a^2;$$
$$\therefore\ CG^2=2BC^2=57120a^2;$$
$$\therefore\ CH^2=57121a^2,$$
$$\therefore\ CH=239a.$$

9. Let ABC be a horizontal section of the two walls, and let ϕ be the inclination of the wall AB to the meridian, so that $\gamma-\phi$ is the inclination of BC to the meridian.

The length of the shadow of the wall AB measured along the meridian is $a\cot\theta$; hence the breadth of the shadow (which is measured at right angles to AB) is $a\cot\theta\sin\phi$,
$$\therefore\ b=a\cot\theta\sin\phi.$$

Similarly,
$$c=a\cot\theta\sin(\gamma-\phi);$$
$$\therefore\ c=a\cot\theta\,(\sin\gamma\cos\phi-\cos\gamma\sin\phi)$$
$$=a\cot\theta\sin\gamma\cos\phi-b\cos\gamma;$$
$$\therefore\ c+b\cos\gamma=a\cot\theta\sin\gamma\cos\phi.$$

Also
$$b\sin\gamma=a\cot\theta\sin\gamma\sin\phi.$$

By squaring and adding, we have
$$c^2+2bc\cos\gamma+b^2=a^2\cot^2\theta\sin^2\gamma.$$

MISCELLANEOUS EXAMPLES. K.

1. If x is the number of degrees in the vertical angle $x+12x+12x=180$, whence $x=\frac{180}{25}=7{\cdot}2$. Thus the angle is $7°\,12'$.

Again, the number of grades $=\frac{200}{180}\times\frac{180}{25}=8$.

2. We have
$$\frac{\alpha}{4}=\frac{\beta}{5}=\frac{\gamma}{6}=\frac{\alpha+\beta+\gamma}{15}=\frac{\pi}{15},$$

whence
$$\alpha=\frac{4\pi}{15},\quad \beta=\frac{\pi}{3},\quad \gamma=\frac{2\pi}{5}.$$

3. The first side $=\left(\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}\right)\div\left(\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)$
$$=(\cos^2A-\sin^2A)\div(\cos^2A+\sin^2A)$$
$$=\cos^2A-\sin^2A=(1-\sin^2A)-\sin^2A$$
$$=1-2\sin^2A.$$

4. $\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{25}{169}} = \frac{12}{13}$;

$$\therefore \tan A + \sec A = \frac{5}{12} + \frac{13}{12} = \frac{3}{2}.$$

5. Let $AB=15$, $AC=30$, then since $\cos 60° = \frac{1}{2}$, it is easy to see that CB is at right angles to BA.

$$\therefore CB = CA \sin 60° = 30 \times \frac{\sqrt{3}}{2} = 25{\cdot}98.$$

6. Let AB be the tower, BD the cliff, and C the point of observation; then if $BD=x$ ft., $CD=y$ ft., we have $50+x=y\tan\alpha$, $x=y\tan\beta$.

By division $$1+\frac{50}{x} = \frac{\tan\alpha}{\tan\beta} = \frac{1260}{1185} = \frac{84}{79}.$$

$$\therefore \frac{50}{x} = \frac{5}{79}, \text{ and } x = 790.$$

7. If x ft. be the length of the arc, $\frac{x}{30}$ = radian measure of 10°; whence

$$x = 30 \times \frac{\pi}{180} \times 10 = 5{\cdot}236.$$

8. Here $\tan\alpha = \frac{8}{15}$; $\therefore \sin\alpha = \frac{8}{17}$, $\cos\alpha = \frac{15}{17}$.

9. Here $$4\sin^2\theta - (2+2\sqrt{3})\sin\theta + \sqrt{3} = 0;$$

whence $$(2\sin\theta - \sqrt{3})(2\sin\theta - 1) = 0.$$

10. The expression $= \frac{5\tan\alpha + 7}{6 - 3\tan\alpha} = \frac{5\times 4 + 7\times 15}{6\times 15 - 3\times 4}$

$$= \frac{5}{3} \times \frac{25}{26} = \frac{125}{78}.$$

11. First side $= 1 + 2(\sin A + \cos A) + (\sin A + \cos A)^2$

$= 1 + 2(\sin A + \cos A) + 1 + 2\sin A \cos A$

$= 2 + 2(\sin A + \cos A) + 2\sin A \cos A$

$= 2(1+\sin A)(1+\cos A)$.

12. The expression $= \sec^2 A(2 - \sec^2 A) - \operatorname{cosec}^2 A(2 - \operatorname{cosec}^2 A)$

$= (1+\tan^2 A)(1-\tan^2 A) - (1+\cot^2 A)(1-\cot^2 A)$

$= (1-\tan^4 A) - (1-\cot^4 A)$

$= (1-\tan^4 A) - \left(1 - \frac{1}{\tan^4 A}\right)$

$= (1-\tan^4 A)\left(1 + \frac{1}{\tan^4 A}\right) = \frac{1-\tan^8 A}{\tan^4 A}$.

13. $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{\sin\alpha - \cos\alpha}{\sqrt{(\sin\alpha-\cos\alpha)^2+(\sin\alpha+\cos\alpha)^2}}$

$= \dfrac{\sin\alpha-\cos\alpha}{\sqrt{2(\sin^2\alpha+\cos^2\alpha)}} = \dfrac{\sin\alpha-\cos\alpha}{\sqrt{2}}$.

14. First side $= \left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{2}\right) \div \left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\right) = \dfrac{\sqrt{2}-1}{\sqrt{2}+1} = \dfrac{\sec 45° - \tan 45°}{\operatorname{cosec} 45° + \cot 45°}$.

15. Since 9 degrees = 10 grades

$$1' = \frac{1}{60} \times \frac{10}{9} \times 100 \times 100 \times 1'' = \frac{10^4}{54} \times 1''.$$

16. (2) Second side $= \left(1 - \dfrac{\sin\theta}{\cos\theta}\right)^2 \div \left(1 - \dfrac{\cos\theta}{\sin\theta}\right)^2$

$= \left(\dfrac{\cos\theta - \sin\theta}{\cos\theta}\right)^2 \div \left(\dfrac{\sin\theta - \cos\theta}{\sin\theta}\right)^2$

$= \dfrac{\sec^2\theta}{\operatorname{cosec}^2\theta} = \dfrac{1+\tan^2\theta}{1+\cot^2\theta}$.

17. (1) $$\sin\theta + \frac{1}{\sin\theta} = \frac{3}{\sqrt{2}};$$

$$\therefore \sin^2\theta - \frac{3}{\sqrt{2}}\sin\theta + 1 = 0,$$

$$\sqrt{2}\sin^2\theta - 3\sin\theta + \sqrt{2} = 0,$$

$$(\sqrt{2}\sin\theta - 1)(\sin\theta - \sqrt{2}) = 0;$$

whence $\sin\theta = \dfrac{1}{\sqrt{2}}$, and $\theta = 45°$, the other value being impossible.

(2) $$\cos\theta + \frac{1}{\cos\theta} = \frac{5}{2};$$

$$\therefore 2\cos^2\theta - 5\cos\theta + 2 = 0,$$

$$(2\cos\theta - 1)(\cos\theta - 2) = 0;$$

whence $\cos\theta = \dfrac{1}{2}$, and $\theta = 60°$, the other value being impossible.

18. Radian measure of $56° = \dfrac{\pi}{180} \times 56 = \dfrac{22}{7} \times \dfrac{56}{180} = \dfrac{44}{45}$.

The arc traversed in $36'' = \dfrac{36}{60} \times \dfrac{10}{60} \times 1760$ yards $= 176$ yards.

$\therefore$ if d be the number of yards in the diameter,

$$\frac{176}{d} = \frac{22}{45}; \quad \therefore d = 360.$$

19. See Art. 35.

20. (1) First side $= 1 \times \left(\frac{\sin^2 A}{\cos^2 A} - 1\right) = \frac{\sin^2 A - \cos^2 A}{\cos^2 A}$

$= \sec^2 A\,(\sin^2 A - \cos^2 A) =$ second side.

(2) Second side $= \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \beta} - \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \alpha}$

$= \sin^2 \alpha - \sin^2 \beta$

$=$ First side. [Examples III. *b*, 34.]

21. We have $\cos B = \frac{a}{c} = \cdot 405$; and $a + c = 281$.

$\therefore\ c(1 + \cdot 405) = 281$; whence $c = 200$, $a = 81$.

Also $b = \sqrt{c^2 - a^2} = \sqrt{33439} = 183$ nearly.

22. The radian measure $= \frac{\text{arc}}{\text{radius}} = \frac{495}{3 \times 1760} = \frac{3}{32} = \cdot 09375$.

With this as unit a right angle would be $\frac{\pi}{2} \div \frac{3}{32} = 16 \cdot 7552$.

23. (1) First side $= \sin\theta \cos\theta \left\{\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right\} = \cos^2\theta + \sin^2\theta = 1$.

(2) First side $= \frac{\cot\theta}{\sec\theta} \times \frac{\cot^2\theta}{\operatorname{cosec}\theta} \times \frac{\cos\theta}{\sin^3\theta}$

$= \cot^3\theta \times \frac{\cos\theta}{\sin\theta} \times \frac{\cos\theta}{\sin\theta} = \cot^5\theta$.

24. If O be the point of observation, $\angle BOA = 45°$, and the line drawn from B perpendicular to OA bisects it at a point A'. Then

$$OA : OB = 2OA' : OB = 2\cos 45° = \sqrt{2} : 1.$$

25. (1) First side $= (\sin^2 A + \operatorname{cosec}^2 A - 2) + (\cos^2 A + \sec^2 A - 2)$

$= (\sin^2 A + \cos^2 A) + (\operatorname{cosec}^2 A - 1) + (\sec^2 A - 1) - 2$

$= \cot^2 A + \tan^2 A - 1$.

(2) First side $= 3\cot^2\theta - 10\cot\theta + 3 = 3\operatorname{cosec}^2\theta - 10\cot\theta$.

27. The expression $= \frac{2 - \cot A}{2 + 3\cot A} = \frac{2 - \frac{9}{2}}{2 + \frac{27}{2}} = -\frac{5}{31}$.

28. $2\cos\theta\cot\theta+1-\cot\theta-2\cos\theta=0;$

$\therefore\ 2\cos\theta(\cot\theta-1)-(\cot\theta-1)=0;$

$\therefore\ (2\cos\theta-1)(\cot\theta-1)=0;$

$\therefore\ \theta=60°,$ or $45°.$

29. If x inches be the length of the arc, $\frac{x}{6}=\frac{\pi}{180}\times\frac{1217}{60}.$

In the second circle $\theta=\frac{\pi}{180}\times\frac{1217}{10}\times\frac{1}{8};$

$\therefore$ sexagesimal measure of $\theta=\frac{1217}{10\times8}=15°\,12'\,45''.$

30. Take the figure of the Example on p. 41, and let

$PT=x$ yards, $RT=y$ yards.

Then $x=y+110$, since $\angle PQT=45°.$

Also $y=x\cot 60°=\frac{x}{\sqrt{3}}.$

$\therefore\ x\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)=110;$

$\therefore\ x=55(3+\sqrt{3})=260{\cdot}26.$

31. First side $=\frac{1+\cos A}{1-\cos A}-\frac{1-\cos A}{1+\cos A}-4\cot^2 A$

$=\frac{(1+\cos A)^2-(1-\cos A)^2}{(1-\cos A)(1+\cos A)}-4\cot^2 A$

$=\frac{2\,.\,2\cos A}{\sin^2 A}-\frac{4\cos^2 A}{\sin^2 A}=\frac{4\cos A(1-\cos A)}{1-\cos^2 A}$

$=\frac{4\cos A}{1+\cos A}=\frac{4}{1+\sec A}.$

32. (1) $8(1-\cos^2\theta)-2\cos\theta=5;$

$8\cos^2\theta+2\cos\theta-3=0;$

$(2\cos\theta-1)(4\cos\theta+3)=0;$

whence $\theta=60°$; or $\cos\theta=-\frac{3}{4}.$

(2) $5\tan^2 x-(1+\tan^2 x)=11;$

$4\tan^2 x=12;$

$\tan x=\pm\sqrt{3}.$

From the first of these values $x=60°.$

33. Here $\theta = \dfrac{\text{arc}}{\text{radius}} = \dfrac{11}{5 \times 12}$;

$$\therefore \text{ the angle in degrees} = \frac{180}{\pi} \times \frac{11}{60} = \frac{180}{22} \times \frac{11 \times 7}{60} = 10\tfrac{1}{2}^\circ.$$

35. See Art. 16.

(2) Here $\cos\theta = \dfrac{2ab}{a^2+b^2}$.

Now since $(a-b)^2$ is a positive quantity, $a^2+b^2 > 2ab$; $\therefore \cos\theta < 1$, which is possible.

36. Let ACB be the hill, A being the summit and C a point halfway down. Draw AD, CE perpendicular to the horizontal line through the object O. Then $AD = 2CE$, and $BD = 2BE$.

Now $OB + BE = CE \cot\beta$, and $OB + 2BE = 2CE \cot\alpha$;

therefore, by subtraction, $\dfrac{OB+2BE}{CE} - \dfrac{OB+BE}{CE} = 2\cot\alpha - \cot\beta$;

that is, $\dfrac{BE}{CE} = 2\cot\alpha - \cot\beta$; also $\dfrac{BE}{CE} = \cot\theta$.

37. From the figure in Ex. 2, Art. 46, we have

$$\cos B = \frac{a}{c} = \frac{25\sqrt{2}}{50} = \frac{1}{\sqrt{2}}; \quad \therefore B = 45^\circ;$$

$$\therefore b = c \sin 45^\circ = 50 \times \frac{1}{\sqrt{2}} = 25\sqrt{2}.$$

If p be the perpendicular from C, $p = a \sin 45^\circ = 25$.

40. Since $\tan\theta = \pm 1$, the angles will be those coterminal with 45°, 135°, 225°, 315°.

41. Take the figure of Example II. on p. 43.

Let $AE = x$, $CE = y$; then

$$\frac{x}{y} = \cdot 965; \quad \frac{x+42}{y} = 1\cdot 6.$$

$$\therefore \frac{x+42}{x} = \frac{1600}{965} = 1 + \frac{635}{965};$$

whence $x = 42 \times \dfrac{965}{635} = 63$, approximately;

$$\therefore AB = 63 + 42 = 105.$$

42. We have $\tan 15° = \dfrac{\sin 30°}{1+\cos 30°} = \dfrac{1}{2} \Big/ \left(1+\dfrac{\sqrt{3}}{2}\right) = \dfrac{1}{2+\sqrt{3}} = 2-\sqrt{3}.$

Similarly $\tan 75° = 2+\sqrt{3}.$

Again, $1+\tan^2\theta = 4\tan\theta,$

$$\tan^2\theta - 4\tan\theta + 1 = 0;$$

$$\therefore \tan\theta = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3};$$

$$\therefore \theta = 75°, \text{ or } 15°.$$

43. The first side

$$= 1 + \sec\theta + \tan\theta + \operatorname{cosec}\theta + \sec\theta\operatorname{cosec}\theta + \operatorname{cosec}\theta\tan\theta + \cot\theta + \sec\theta\cot\theta + 1$$

$$= 2 + \sec\theta + \tan\theta + \operatorname{cosec}\theta + \sec\theta\operatorname{cosec}\theta + \sec\theta + \cot\theta + \operatorname{cosec}\theta$$

$$= 2(1 + \sec\theta + \operatorname{cosec}\theta) + \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta\cos\theta}$$

$$= 2(1 + \sec\theta + \operatorname{cosec}\theta) + \frac{2(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta}$$

$$= 2(1 + \sec\theta + \operatorname{cosec}\theta + \tan\theta + \cot\theta).$$

44. See figure and notation of Art. 25.

45. We have $2(1-\sin^2\theta) = 1 + \sin\theta;$

$$\therefore (1+\sin\theta)(1-2\sin\theta) = 0;$$

$$\therefore \sin\theta = -1, \text{ or } \sin\theta = \frac{1}{2};$$

whence $\theta = 30°, 150°, 270°.$

46 $\sin(270° + A) = -\sin(90° + A) = -\cos A.$

But $\cos A = \pm\sqrt{1-\sin^2 A} = \pm\cdot 8;$

$$\therefore \sin(270° + A) = \pm\cdot 8.$$

47. See Art. 113.

48. The expression $= \dfrac{2\sin 2A \sin A}{2\sin A\cos 2A} = \tan 2A.$ See Art. 89.

49. $\tan A = \sqrt{\sec^2 A - 1} = \pm\sqrt{\dfrac{4}{3} - 1} = \pm\dfrac{1}{\sqrt{3}}.$

The angle is coterminal with 150° or 210°, and the tangents of these angles are equal but opposite in sign.

50. Take the figure of Example on p. 41, and let

$$PT = x \text{ yards}, \quad \angle PQT = 45°, \quad \angle PRT = 60°.$$

Also $QR = 1760$, and $QT = PT$;

$$\therefore \frac{x}{x - 1760} = \tan 69° = \sqrt{3};$$

$$\therefore x(\sqrt{3} - 1) = 1760\sqrt{3};$$

$$\therefore x = 880(3 + \sqrt{3}) = 4164{\cdot}16.$$

52. (1) First side $= \sin^2\alpha(\sin^2\alpha + 2\cos^2\alpha)$

$$= (1 - \cos^2\alpha)(1 + \cos^2\alpha) = 1 - \cos^4\alpha.$$

(2) First side $= \sec 2\left(\frac{\pi}{4} - \theta\right)$ [Art. 124]

$$= \operatorname{cosec} 2\theta.$$

(3) $\cos 10° + \sin 40° = \cos 10° + \cos 50° = 2\cos 30°\cos 20°$

$$= \sqrt{3}\sin 70°.$$

53. The expression $= \frac{a\tan\theta - b}{a\tan\theta + b} = \frac{a^2 - b^2}{a^2 + b^2}$.

54. Multiply all through by $\cos 18°$, then we have to prove that

$$4\cos^2 18° - 3 = 2\sin 18°.$$

First side $= 4\left\{1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2\right\} - 3 = \frac{10 + 2\sqrt{5}}{4} - 3$

$$= \frac{\sqrt{5} - 1}{2} = 2\sin 18°.$$

56. Here $\theta = \frac{\text{arc}}{\text{radius}} = \frac{20 \times 10}{60 \times 60} \div \frac{1}{2} = \frac{1}{9}$;

$$\therefore D = \frac{1}{9} \times 180 \times \frac{7}{22} = \frac{70}{11} = 6\tfrac{4}{11}°.$$

57. The expression $= \frac{2 - 3\cot\alpha}{4 - 9\tan\alpha}$, where $\tan\alpha = \frac{12}{5}$,

$$= \frac{2 - 3 \times \frac{5}{12}}{4 - 9 \times \frac{12}{5}} = \frac{3}{4} \div \left(-\frac{88}{5}\right)$$

$$= -\frac{15}{352}.$$

58. (2) First side $= \dfrac{2 \sin \frac{\pi}{4} \sin\theta}{2 \cos \frac{2\pi}{3} \sin\theta} + \sqrt{2}$

$$= \frac{1}{\sqrt{2}} \left| \left(-\frac{1}{2} \right) + \sqrt{2} = 0. \right.$$

59. (1) The expression $= \dfrac{\cos A \cos C + (-\cos C)(-\cos A)}{\cos A \sin C - \sin C (-\cos A)}$

$$= \frac{2 \cos A \cos C}{2 \cos A \sin C} = \cot C.$$

(2) The expression $= \dfrac{\sin A \cos A + \text{two similar terms}}{\sin A \sin B \sin C}$

$$= \frac{1}{2} \left(\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A \sin B \sin C} \right)$$

$$= 2. \quad \text{[Art. 135, Ex. 1.]}$$

60. Let ABC be the horizontal equilateral triangle, and let PQ be the flagstaff. Then since each side subtends an angle of 60° at P, the top of the flagstaff, the triangles PCB, PBA, PAC are equilateral.

Let x be a side of $\triangle ABC$;

then $$AQ = \frac{x}{2 \sec 30°} = \frac{x}{\sqrt{3}}.$$

Then from $\triangle PAQ$,

$$PA^2 = AQ^2 + QP^2,$$

$$x^2 = \frac{x^2}{3} + 10000;$$

whence $$x^2 = 15000, \text{ or } x = 50\sqrt{6}.$$

61. The first expression $= \sin\theta + \cos\theta + \sin^2\theta + \cos^2\theta$

$$= \sin\theta + \cos\theta + 1.$$

Similarly, the second expression $= \sin\theta + \cos\theta - 1$;

$\therefore$ the product $= (\sin\theta + \cos\theta)^2 - 1 = 2 \sin\theta \cos\theta = \sin 2\theta.$

62. Second side $= \sin^2 \dfrac{\theta+\phi}{2} + \cos^2 \dfrac{\theta-\phi}{2} - 2 \sin \dfrac{\theta+\phi}{2} \cos \dfrac{\theta-\phi}{2}$

$$= \frac{1}{2} \{1 - \cos(\theta+\phi) + 1 + \cos(\theta-\phi)\} - \sin\theta - \sin\phi.$$

$$= 1 - \sin\theta - \sin\phi + \frac{1}{2} \{\cos(\theta-\phi) - \cos(\theta+\phi)\}$$

$$= 1 - \sin\theta - \sin\phi + \sin\theta \sin\phi$$

$$= (1 - \sin\theta)(1 - \sin\phi).$$

63. The expression $= \dfrac{2\cos\alpha\sin\theta}{2\sin\beta\sin\theta} = \dfrac{\cos\alpha}{\sin\beta}$, which is independent of θ.

64. First side $= \sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2} + \ldots + \ldots$

$$= \frac{1}{2}\{(\sin B + \sin C) + \ldots + \ldots\}$$

$$= \sin A + \sin B + \sin C.$$

65. We have $$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}};$$

$$2\sin^2\frac{\theta}{2} = \left(2\cos^2\frac{\theta}{2} - 1\right)^2;$$

$$\therefore\ 4\cos^4\frac{\theta}{2} - 2\cos\frac{\theta}{2} - 1 = 0;$$

$$\therefore\ \cos^2\frac{\theta}{2} = \frac{2\pm\sqrt{20}}{8} = \frac{1\pm\sqrt{5}}{4};$$

$$\therefore\ \cos^2\frac{\theta}{2} = \cos 36^\circ, \text{ or } \cos 108^\circ.$$

66. Draw ZW perpendicular to XY and let $ZW = x$. Then $\triangle XZY$ is right-angled at Z.

And $$XZ = XY\cos 30^\circ = 100\sqrt{3} \text{ yds.}$$

Again, from $\triangle WXZ$, $WZ = XZ\sin 30^\circ = 50\sqrt{3} = 86{\cdot}6$ yds.

67. We have $32\pi \times 1000 = \dfrac{5585}{2} \times 3 \times 12$; whence $\pi = 3{\cdot}141$, approximately.

68. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{1-\cos 2\alpha}{2} + \ldots + \ldots$

$$= \frac{3}{2} - \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\gamma)$$

$$= \frac{3}{2} - \frac{1}{2}\{2\cos(\alpha+\beta)\cos(\alpha-\beta) + 1 - 2\sin^2\gamma\}$$

$$= 1 - \{\sin\gamma\cos(\alpha-\beta) - \sin^2\gamma\}$$

$$= 1 - \sin\gamma\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\}$$

$$= 1 - 2\sin\alpha\sin\beta\sin\gamma.$$

69. (1) First side $= \left(\frac{\sin A}{\cos A} + \frac{\sin 2A}{\cos 2A}\right)(2 \cos 2A \cos A)$

$= 2(\sin 2A \cos A + \cos 2A \sin A) = 2 \sin 3A.$

(2) Multiply all through by 32; then

Second side $= 2 + \cos 2A - 2 \cos 4A - \cos 6A$

$= 2(1 - \cos 4A) + 2 \sin 4A \sin 2A$

$= 4 \sin^2 2A + 4 \sin^2 2A \cos 2A$

$= 4 \sin^2 2A (1 + \cos 2A)$

$= 16 \sin^2 A \cos^2 A \,.\, 2 \cos^2 A$

$= 32 \sin^2 A \cos^4 A.$

70. The expression $= \frac{2 \cos 13a \sin 10a}{2 \sin 10a \cos 6a} = \frac{\cos 13a}{\cos 6a}$

$= \frac{\cos 13a}{\cos(\pi - 13a)} = -1.$

71. We have $\cot(A+B) = 1$. Therefore

$\cot A \cot B - 1 = \cot A + \cot B;$

$\therefore\ 2 \cot A \cot B = 1 + \cot A + \cot B + \cot A \cot B$

$= (1 + \cot A)(1 + \cot B);$

$\therefore\ \frac{\cot A}{1 + \cot A} \cdot \frac{\cot B}{1 + \cot B} = \frac{1}{2}.$

72. For the first part see XI. d. Ex. 15.

Then $\tan\theta = \cot\theta - 2\cot 2\theta,$

$2 \tan 2\theta = 2 \cot 2\theta - 4 \cot 4\theta,$

$4 \tan 4\theta = 4 \cot 4\theta - 8 \cot 8\theta;$

$\therefore$ by addition

$\tan\theta + 2 \tan 2\theta + 4 \tan 4\theta = \cot\theta - 8 \cot 8\theta.$

73. The expression $= 1 - \frac{\sin^3\theta}{\sin\theta + \cos\theta} - \frac{\cos^3\theta}{\sin\theta + \cos\theta}$

$= 1 - \frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$

$= 1 - (\sin^2\theta + \cos^2\theta - \sin\theta \cos\theta)$

$= \sin\theta \cos\theta = \frac{1}{2} \sin 2\theta.$

74. We have $x = 3\sin A - (3\sin A - 4\sin^3 A) = 4\sin^3 A$,

$$y = 4\cos^3 A - 3\cos A + 3\cos A = 4\cos^3 A;$$

$$\therefore \left(\frac{x}{4}\right)^{\frac{2}{3}} + \left(\frac{y}{4}\right)^{\frac{2}{3}} = \sin^2 A + \cos^2 A = 1;$$

$$\therefore x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}.$$

75. Let PQ, RS be the flagstaffs of lengths x, y feet respectively. Then $QABS$ is a straight line, such that $AB = 30$ ft., $\angle PAQ = 60°$, $\angle RAS = 30°$, $\angle PBQ = 45°$, $\angle RBS = 60°$. Let $AQ = a$, $BS = b$. Then since

$$\angle PBQ = 45°, \quad BQ = QP = x;$$

$$\therefore a = x - 30.$$

From $\triangle ARS$, we have $AS = RS\cot 30° = y\sqrt{3}$;

$$\therefore b = y\sqrt{3} - 30.$$

From $\triangle APQ$, $PQ = AQ\tan 60°$;

$$\therefore x = (x - 30)\sqrt{3};$$

whence $$x = 15(3 + \sqrt{3}).$$

From $\triangle BRS$, $RS = BS\tan 60°$;

$$\therefore y = \sqrt{3}(y\sqrt{3} - 30);$$

whence $$y = 15\sqrt{3}.$$

Again,
$$\begin{aligned} QS &= a + b + 30 \\ &= x + y\sqrt{3} - 30 \\ &= 45 + 15\sqrt{3} + 45 - 30 \\ &= 60 + 15\sqrt{3}. \end{aligned}$$

76. (1) First side

$$\begin{aligned} &= \frac{1}{2}(1 + \cos 2A) + \frac{1}{2}(1 + \cos 2B) - 2\cos A\cos B\cos(A + B) \\ &= 1 + \cos(A + B)\cos(A - B) - 2\cos A\cos B\cos(A + B) \\ &= 1 + \cos(A + B)\{\sin A\sin B - \cos A\cos B\} \\ &= 1 - \cos^2(A + B) = \sin^2(A + B). \end{aligned}$$

(2) First side

$$\begin{aligned} &= 2(\sin 5A - \sin A) - (\sin 3A + \sin A) \\ &= 4\cos 3A\sin 2A - 2\sin 2A\cos A \\ &= 2\sin 2A(2\cos 3A - \cos A) \\ &= 4\sin A\cos A(8\cos^3 A - 7\cos A) \\ &= 4\sin A\cos^2 A\{8(1 - \sin^2 A) - 7\} \\ &= 4\sin A\cos^2 A(1 - 8\sin^2 A). \end{aligned}$$

77. If r is the radius of the circle, and AB one side of the square, we have $2\pi r = 3$, and

$$AB = 2r \sin 45° = \frac{3\sqrt{2}}{2\pi}$$

$$= \frac{3}{2} \times 1{\cdot}4142 \times {\cdot}3183$$

$$= {\cdot}6752 \text{ feet}$$

$$= 8{\cdot}10 \text{ inches.}$$

78. Here $AB = 2r \sin 54°$, $BC = 2r \sin 30°$, $CD = 2r \sin 18°$. And it remains to prove that

$$\sin 54° = \sin 30° + \sin 18°.$$

[See Examples XI. *e*. 9.]

79. First side $= (2+\sqrt{3}) + (2-\sqrt{3}) - 1 - 2 = 1$.

80. We have, by addition, $\cot\theta = 2(m+n)$.

Also, by subtraction, $\cos\theta = 2(m-n)$.

$$\therefore\ 4(m^2-n^2) = \frac{\cos^2\theta}{\sin\theta};$$

$$\therefore\ 16(m^2-n^2)^2 = \frac{\cos^4\theta}{\sin^2\theta} = \cot^2\theta \times \cos^2\theta$$

$$= \cot^2\theta\,(1-\sin^2\theta)$$

$$= 16mn.$$

81. (1) First side $= \frac{1}{2}[\sin(2\beta+\alpha) + \sin\alpha - \sin(2\gamma+\alpha) - \sin\alpha]$

$$= \frac{1}{2}[2\cos(\alpha+\beta+\gamma)\sin(\beta-\gamma)].$$

(2) First side $= \left(\frac{\sin 2A}{\cos 2A} - \frac{\sin A}{\cos A}\right)\left(\frac{1}{\cos A} + \frac{1}{\cos 3A}\right)$

$$= \frac{\sin A}{\cos 2A \cos A} \cdot \frac{\cos 3A + \cos A}{\cos A \cos 3A}$$

$$= \frac{2\sin A \cos 2A \cos A}{\cos 2A \cos^2 A \cos 3A} = 2\sin A \sec A \sec 3A.$$

82. $2\cos 6° \cos 66° = \cos 72° + \cos 60° = \sin 18° + \frac{1}{2}$

$$= \frac{\sqrt{5}-1}{4} + \frac{1}{2} = \frac{\sqrt{5}+1}{4}.$$

$$2\cos 42^\circ \cos 78^\circ = \cos 120^\circ + \cos 36^\circ$$
$$= -\frac{1}{2} + \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}-1}{4}.$$
$$\therefore\ 4\cos 6^\circ \cos 66^\circ \cos 42^\circ \cos 78^\circ = \frac{1}{4}.$$

83. Put $2A = 45^\circ$.

84. The distance required is evidently equal to $10\tan 22\frac{1}{2}^\circ$
$$= \frac{10}{\sqrt{2}+1} = 10(\sqrt{2}-1) = 4{\cdot}14 \text{ miles}.$$

85. (1) Separate each term into the difference of two cosines.

(2) Second side $= \dfrac{\sin\theta + 2\sin\theta\cos\theta}{\cos\theta + 2\cos^2\theta}$
$$= \frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)} = \tan\theta.$$

86. First side $= 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2} + 2\cos^2\dfrac{\gamma}{2} - 1$
$$= 2\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} - 1$$
$$= 2\cos\frac{\gamma}{2}\left\{\cos\frac{\alpha-\beta}{2} + \cos\frac{\gamma}{2}\right\} - 1$$
$$= 2\cos\frac{\gamma}{2}\left\{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}\right\} - 1$$
$$= 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - 1.$$

87. Put $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$, then

first side $= \dfrac{k^2(\sin^2 B - \sin^2 C)}{k\sin A}\cos A + \ldots + \ldots$
$$= k\left\{\frac{\sin(B+C)\sin(B-C)}{\sin A}\cos A + \ldots + \ldots\right\}$$
$$= k\{\sin(B-C)\cos A + \ldots + \ldots\}$$
$$= -k\{\sin(B-C)\cos(B+C) + \ldots + \ldots\}$$
$$= -\frac{k}{2}\{(\sin 2B - \sin 2C) + \ldots + \ldots\}$$
$$= 0.$$

88. First side $= a\cos 2\theta + b\sin 2\theta$

$= a(1-2\sin^2\theta) + 2b\sin\theta\cos\theta$

$= a + 2\sin\theta(b\cos\theta - a\sin\theta)$

$= a$, since $b\cos\theta = a\sin\theta$.

89. Let $\log_a b = x$, so that $a^x = b$,

$\log_b c = y$, ... $b^y = c$,

$\log_c a = z$, ... $c^z = a$.

Then we have $a = c^z = b^{yz} = a^{xyz}$;

$\therefore\ xyz = 1$, or $\log_a b \log_b c \log_c a = 1$.

We have $\log 8 = \log 2^3 = 3\log 2$; whence $\log 2 = \cdot 30103$;

$$\log 2\cdot 4 = \log\left(\frac{3\times 8}{10}\right) = \log 3 + \log 8 - 1$$

$$= \bar{1}\cdot 47712 + \cdot 90309 = \cdot 38021.$$

$$\log 5400 = 2 + \log 2 + 3\log 3$$

$$= 2\cdot 30103 + 1\cdot 43136 = 3\cdot 73239.$$

$$L\tan 30° = 10 + \log\frac{1}{\sqrt{3}} = 10 - \frac{1}{2}\log 3$$

$$= 9\cdot 76144.$$

90. $\cot(A+B) = \cot(90°-C) = \tan C = \dfrac{1}{\cot C}$;

$$\therefore\ \frac{\cot A\cot B - 1}{\cot A + \cot B} = \frac{1}{\cot C};$$

whence by multiplying up and rearranging we obtain the required result.

For the second part, put $A = 15°$, $B = 30°$, $C = 45°$.

91. First side $= (1+\sin 2A)^2 + \cos^2 2A + 2\cos 2A(1+\sin 2A)$

$= (1+\sin 2A)^2 + (1-\sin^2 2A) + 2\cos 2A(1+\sin 2A)$

$= (1+\sin 2A)\{1+\sin 2A + 1 - \sin 2A + 2\cos 2A\}$

$= 2(1+\cos 2A)(1+\sin 2A) = 4\cos^2 A(1+\sin 2A)$.

92. See Art. 150.

93. We have to prove that $\sin 9°\sin 81° = \sin 12°\sin 48°$.

$$\text{First side} = \frac{1}{2}(\cos 72° - \cos 90°)$$

$$= \frac{1}{2}\sin 18° = \frac{\sqrt{5}-1}{8}.$$

$$\text{Second side} = \frac{1}{2}(\cos 36° - \cos 60°)$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right) = \frac{\sqrt{5}-1}{8}.$$

94. See Art. 136, Ex. 2.

95. $L \sin\theta > L \sin 27° 45'$ by $\frac{1742}{2400} \times 60''$;

whence $$\theta = 27° 45' 44''.$$

96. By a well-known algebraical formula,

$$x^3 + y^3 + z^3 = 3xyz,$$

when $$x + y + z = 0;$$

therefore we have

$$\cos^3 A + \cos^3 B + \cos^3 C = 3 \cos A \cos B \cos C.$$

Substituting $\frac{1}{4}(\cos 3A + 3\cos A)$ for $\cos^3 A$, and similar results for $\cos^3 B$, $\cos^3 C$, we have

$$\frac{1}{4}(\cos 3A + \cos 3B + \cos 3C) + \frac{3}{4}(\cos A + \cos B + \cos C) = 3 \cos A \cos B \cos C,$$

whence the required result follows at once, since

$$\cos A + \cos B + \cos C = 0.$$

97. $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{49}{625}} = \pm\frac{24}{25}$; but since A lies between 270° and 360°, we must reject the negative value; thus $\cos A = \frac{24}{25}$.

Hence $\sin 2A = 2 \sin A \cos A = 2\left(-\frac{7}{25}\right)\left(\frac{24}{25}\right)$

$$= -\frac{336}{625}.$$

Also $\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A} = \operatorname{cosec} A - \cot A$

$$= -\frac{25}{7} + \frac{24}{7} = -\frac{1}{7}.$$

98. We have $$\frac{2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \left(\frac{\sin\theta+\cos\theta}{\sin\theta}\right)^2;$$

$$8\cos^3\frac{\theta}{2}\sin\frac{\theta}{2} = 1+\sin 2\theta,$$

$$4\cos^2\frac{\theta}{2}\sin\theta = 1+\sin 2\theta,$$

$$2\sin\theta(1+\cos\theta) = 1+\sin 2\theta;$$

$$\therefore\ 2\sin\theta = 1,$$

or $$\theta = 30°.$$

Again, by putting $\theta = 30°$, we have

$$2\cot 15° = (1+\sqrt{3})^2 = 4+2\sqrt{3}.$$

$$\therefore\ \cot 15° = 2+\sqrt{3}, \text{ and } \tan 15° = 2-\sqrt{3}.$$

99. We have $\log 360 = 2\log 2 + 2\log 3 + 1.$

$$\therefore\ 2\log 3 = \log 360 - 2\log 2 - 1$$

$$= 1{\cdot}5563025 - {\cdot}6020600$$

$$= {\cdot}9542425;$$

$$\therefore\ \log 3 = {\cdot}4771213.$$

Now $$\log {\cdot}04 = \log 4 - 2 = 2\log 2 - 2 = \bar{2}{\cdot}60206.$$

$$\log 24 = 3\log 2 + \log 3 = {\cdot}90309 + {\cdot}4771213$$

$$= 1{\cdot}3802113.$$

$$\log \dot{6} = \log\frac{2}{3} = \log 2 - \log 3$$

$$= {\cdot}30103 - {\cdot}4771213 = \bar{1}{\cdot}8239087.$$

Again let $\log_2 30 = x$, so that $2^x = 30$.

$$\therefore\ x\log 2 = \log 30 = 1+\log 3;$$

$$\therefore\ x = \frac{1+\log 3}{\log 2} = \frac{1{\cdot}4771213}{{\cdot}30103} = 4{\cdot}90689.$$

100. This follows at once from Art. 134, Ex. 5.

101. $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}=\dfrac{x(x+x^{-1}+1)-1}{(x+x^{-1}+1)^{\frac{1}{2}}(1+x)}=\dfrac{x^2+x}{(1+x)(x+x^{-1}+1)^{\frac{1}{2}}}$

$$=\frac{x}{(x+x^{-1}+1)^{\frac{1}{2}}}=\frac{1}{x^{-1}(x+x^{-1}+1)^{\frac{1}{2}}}=\frac{1}{(x^{-1}+x^{-3}+x^{-2})^{\frac{1}{2}}}$$

$$=\cot\gamma.$$

Therefore $\alpha+\beta=\gamma$.

102. Let A and a be given, and let B be the right angle; then $c=a\cot A$, $b=a\operatorname{cosec}A$, or $b=\sqrt{a^2+c^2}$. Also $C=90°-A$.

If $A=31°\,53'\,26{\cdot}8''$, $a=28$, we have

$$c=28\cot A.$$

$$\log c=\log 28+\log\cot 31°\,53'\,26{\cdot}8'',$$

$$\log 28=1{\cdot}4471580$$

$$\log\cot 31°\,53'=\ {\cdot}2061805$$

decrease for $26{\cdot}8''$ $\qquad$ 1258

$$\therefore\ \log c=1{\cdot}6532127;\ \therefore\ c=45.$$

Again $\qquad b^2=a^2+c^2=2025+784=2809;$

$$\therefore\ b=53;\ \text{also } C=90°-A=58°\,6'\,33{\cdot}2''.$$

104. The greatest angle, C, is opposite to $\sqrt{x^2+xy+y^2}$.

$$\therefore\ \cos C=\frac{x^2+y^2-(x^2+xy+y^2)}{2xy}=-\frac{1}{2};$$

$$\therefore\ C=120°.$$

105. $\cos 3A+\sin 3A=4\cos^3A-3\cos A+3\sin A-4\sin^3A$

$$=4(\cos^3A-\sin^3A)-3(\cos A-\sin A),$$

which is evidently divisible by $\cos A-\sin A$.

See solution to Examples XII. *c*, 27.

106. We have $\qquad \cot\dfrac{C}{2}=\tan\dfrac{A+B}{2}=\dfrac{\frac{5}{6}+\frac{20}{37}}{1-\frac{5}{6}\cdot\frac{20}{37}}=\dfrac{5}{2}.$

$$\therefore\ \tan C=\frac{2\tan\frac{C}{2}}{1-\tan^2\frac{C}{2}}=\frac{2\times\frac{2}{5}}{1-\frac{4}{25}}=\frac{20}{21}.$$

For the second part, it will be sufficient to prove that

$$\sin A+\sin C=2\sin B.$$

Now $\sin A = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} = \dfrac{60}{61}$, on reduction.

Similarly $\sin C = \dfrac{20}{29}$, and $\sin B = \dfrac{1480}{1769}$, whence the result follows.

107. The first factor easily reduces to $2\cot 2\theta$, and the second to $\dfrac{2}{\cos 2\theta}$; whence the product becomes $4\operatorname{cosec} 2\theta$.

108. $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\dfrac{128}{603}}$,

$$\log\tan\frac{A}{2} = \frac{1}{2}\{\log 128 - \log 603\} = \bar{1}\cdot 6634465;$$

whence $\dfrac{A}{2} = 24^\circ\,44'\,16''$, and $A = 49^\circ\,28'\,32''$.

109. $$\tan(A-B) = \frac{\tan A - \dfrac{n\sin A\cos A}{1-n\sin^2 A}}{1+\tan A\left(\dfrac{n\sin A\cos A}{1-n\sin^2 A}\right)} = \frac{\tan A\left(1-\dfrac{n\cos^2 A}{1-n\sin^2 A}\right)}{1+\dfrac{n\sin^2 A}{1-n\sin^2 A}}$$

$$= (1-n)\tan A.$$

110. $\log 200 = 2+\log 2 = 3-\log 5 = 2\cdot 30103.$

$$\log\cdot 025 = 2\log 5 - 3 = \bar{2}\cdot 39794,$$

$$\log\sqrt[3]{62\cdot 5} = \frac{1}{3}(\log 625 - 1) = \frac{1}{3}(4\log 5 - 1)$$

$$= \cdot 598626.$$

$$L\sin 30^\circ = 10 + \log\left(\frac{1}{2}\right) = 10 + \log 5 - 1$$

$$= 9\cdot 69897.$$

$$L\cos 45^\circ = 10 + \log\left(\frac{1}{\sqrt{2}}\right) = 10 + \frac{1}{2}(\log 5 - 1)$$

$$= 9\cdot 849485.$$

111. (1) $$\frac{\cot(A-30^\circ)}{\tan(A+30^\circ)} = \frac{\cos(A-30^\circ)\cos(A+30^\circ)}{\sin(A-30^\circ)\sin(A+30^\circ)}$$

$$= \frac{\cos 2A + \cos 60^\circ}{\cos 60^\circ - \cos 2A} = \frac{2\cos 2A + 1}{1 - 2\cos 2A}$$

$$= \frac{2+\sec 2A}{\sec 2A - 2}.$$

(2) Second side $= 2\left\{\frac{1+\cos 2\alpha}{2}\cdot\frac{1+\cos 2\beta}{2}+\frac{1-\cos 2\alpha}{2}\cdot\frac{1-\cos 2\beta}{2}\right\}$

$$=\frac{1}{2}\{2+2\cos 2\alpha\cos 2\beta\}=1+\cos 2\alpha\cos 2\beta.$$

112. From a diagram it is easily seen that CD is equal to AC, and that from the right-angled triangle ABC,

$$AC=BC\cos 30^\circ=132\frac{\sqrt{3}}{2}$$

$$=66\times\frac{19}{11}=114 \text{ yards.}$$

Also the perp. from A on $BC=AC\sin 30^\circ=57$ yards.

113. Since $\frac{a}{a+b+c}=\frac{\sin A}{\sin A+\sin B+\sin C}$

$$=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}, \qquad \text{[XII. } d\text{. Ex. 3.]}$$

$$\therefore \frac{a\cot\frac{A}{2}+b\cot\frac{B}{2}-c\cot\frac{C}{2}}{a+b+c}=\frac{1}{2}\left\{\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\ldots-\ldots\right\}$$

$$=\frac{1}{2}\left\{\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\right\}.$$

Now $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=\frac{1}{2}\left\{2\cos^2\frac{A}{2}+1+\cos B-1-\cos C\right\}$

$$=\frac{1}{2}\left\{2\cos^2\frac{A}{2}+2\sin\frac{B+C}{2}\sin\frac{C-B}{2}\right\}$$

$$=\cos\frac{A}{2}\left\{\cos\frac{A}{2}+\sin\frac{C-B}{2}\right\}$$

$$=\cos\frac{A}{2}\left\{\sin\frac{B+C}{2}+\sin\frac{C-B}{2}\right\}$$

$$=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Whence the required result easily follows.

114. Here $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{40\times 42}{75\times 77}}$

$$= \frac{2}{\sqrt{5}} = \frac{2\sqrt{2}}{\sqrt{10}},$$

$$\log\cos\frac{A}{2} = \frac{3}{2}\log 2 - \frac{1}{2}\log 10$$

$$= \cdot 4515450 - \cdot 5$$

$$= \bar{1}\cdot 9515450$$

$$\log\cos 26^\circ 34' = \bar{1}\cdot 9515389$$

$$61$$

$$\therefore \frac{A}{2} \text{ is less than } 26^\circ 34' \text{ by } \frac{61}{632}\times 60'';$$

that is,
$$\frac{A}{2} = 26^\circ 33' 54{\cdot}2'',$$

or
$$A = 53^\circ 7' 48''.$$

115. $\sin(A - 90^\circ) = -\sin(90^\circ - A) = -\cos A$

$$= -\sqrt{1-\sin^2 A} = -\sqrt{6\cdot 4}$$

$$= -(\pm\cdot 8);$$

but A is between 90° and 180°, therefore $\sin(A - 90^\circ)$ is positive; that is

$$\sin(A - 90^\circ) = \cdot 8.$$

$$\operatorname{cosec}(270^\circ - A) = \operatorname{cosec}(180^\circ + 90^\circ - A)$$

$$= -\operatorname{cosec}(90^\circ - A)$$

$$= -\left(\frac{1}{\pm\cdot 8}\right)$$

$$= \pm 1\cdot 25;$$

but between the given limits $\operatorname{cosec}(270^\circ - A)$ must be positive, that is, the required value is 1·25.

116. By Art. 168,

$$\log_b c = \frac{\log_a c}{\log_a b}, \quad \log_c d = \frac{\log_a d}{\log_a c};$$

hence the expression on the right $= \log_a b \times \frac{\log_a c}{\log_a b} \times \frac{\log_a d}{\log_a c}$

$$= \log_a d.$$

$$\log_{10} 2 = 1 - \log_{10} 5 = \cdot 30103;$$

$$\log_{10} 8 = 3 \log_{10} 2 = \cdot 90309.$$

$$\log_8 10 = \frac{1}{\log_{10} 8} = 1 \cdot 1073093.$$

$$\log_{10} (\cdot 032)^5 = 5 \log \frac{32}{1000} = 25 \log 2 - 15$$

$$= 7 \cdot 52575 - 15$$

$$= \bar{8} \cdot 52575.$$

117. First side $= \cos (360° + 60° + A) + \cos (60° - A)$

$$= \cos (60° + A) + \cos (60° - A)$$

$$= 2 \cos 60° \cos A = \cos A.$$

For the second part, put $A = 45°$.

118. Write t for $\tan \frac{x}{2}$, then the equation may be written

$$\frac{1 - t^2}{1 + t^2} - \sin \alpha \cot \beta \frac{2t}{1 + t^2} = \cos \alpha;$$

$$\therefore t^2 (1 + \cos \alpha) + 2t \sin \alpha \cot \beta - (1 - \cos \alpha) = 0;$$

$$t^2 + 2 \cot \beta \frac{\sin \alpha}{1 + \cos \alpha} \cdot t - \frac{1 - \cos \alpha}{1 + \cos \alpha} = 0,$$

$$t^2 + 2 \cot \beta \tan \frac{\alpha}{2} t - \tan^2 \frac{\alpha}{2} = 0,$$

$$\left(t + \tan \frac{\alpha}{2} \cot \frac{\beta}{2}\right)\left(t - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}\right) = 0, \qquad \text{[XI. } d\text{. Ex. 15.]}$$

$$\therefore \tan \frac{x}{2} = -\tan \frac{\alpha}{2} \cot \frac{\beta}{2}, \text{ or } \tan \frac{\alpha}{2} \tan \frac{\beta}{2}.$$

119. We have $$\frac{\sin (2A + B)}{\sin B} = \frac{m}{n};$$

$$\therefore \frac{m - n}{m + n} = \frac{\sin (2A + B) - \sin B}{\sin (2A + B) + \sin B}$$

$$= \frac{2 \cos (A + B) \sin A}{2 \sin (A + B) \cos A}$$

$$= \cot (A + B) \tan A.$$

120. Let x feet be the height of the tower;

then
$$\angle ABD = 90° - \angle ADB$$
$$= 45° = \angle ADB;$$
$$\therefore AD = AB = x;$$
$$\therefore AC = x - 17.$$

Now from $\triangle ABC$,
$$x^2 + (x-17)^2 = 53^2;$$
$$\therefore x^2 - 17x - 1260 = 0,$$
$$(x-45)(x+28) = 0;$$
$$\therefore x = 45.$$

Again,
$$\tan ACB = \frac{45}{28};$$

but
$$\tan 31° 48' = \cdot 62\dot{2} = \frac{56}{90} = \frac{28}{45};$$
$$\therefore \angle ACB = 90° - 31° 48'$$
$$= 58° 12'.$$

122.
$$\log 6 = \frac{1}{2}\log 36 = \cdot 778151,$$
$$\log 8 = \log 48 - \log 6 = 1 \cdot 681241 - \cdot 778151$$
$$= \cdot 90309;$$
$$\therefore \log 2 = \frac{1}{3}\log 8 = \cdot 30103;$$
$$\therefore \log 3 = \log 6 - \log 2 = \cdot 477121.$$

Now
$$\log 40 = 1 + 2\log 2 = 1 \cdot 60206.$$
$$\log \sqrt{\frac{2}{15}} = \frac{1}{2}(\log 4 - \log 30)$$
$$= \frac{1}{2}(2\log 2 - \log 3 - 1)$$
$$= \bar{1} \cdot 562469, \text{ on substitution.}$$

123. We have $\tan \frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{9}{10}\cot 72°.$

$$\log \tan \frac{B-C}{2} = 2\log 3 - 1 + \log \cot 72°$$
$$= \bar{1} \cdot 4660186$$
$$\log \tan 16° 18' = \bar{1} \cdot 4660078$$
$$\overline{108}$$

$\frac{108}{4687} \times 60'' = 1''.$

$$\therefore \frac{B-C}{2} = 16° 18' 1'', \quad \frac{B+C}{2} = 18°;$$
$$\therefore B = 34° 18' 1'', \quad C = 1° 41' 59''.$$

124. (1) $\cos A + \cos B \cos C = -\cos \overline{B+C} + \cos B \cos C$

$= \sin B \sin C;$

$\therefore$ Second side $= a^2 \sin B \sin C = b \sin A \,.\, c \sin A$

$= bc \sin^2 A.$

(2) First side $= c\,(b \cos A + a \cos B) + 2ab \cos C$

$= c^2 + 2ab \cos C = a^2 + b^2.$

125. $\tan \frac{\beta - \alpha}{2} = \frac{\tan \frac{\beta}{2} - \tan \frac{\alpha}{2}}{1 + \tan \frac{\beta}{2} \tan \frac{\alpha}{2}} = \frac{3 \tan \frac{\alpha}{2}}{1 + 4 \tan^2 \frac{\alpha}{2}}$

$= \frac{3 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + 4 \sin^2 \frac{\alpha}{2}} = \frac{3 \sin \alpha}{2 \cos^2 \frac{\alpha}{2} + 8 \sin^2 \frac{\alpha}{2}}$

$= \frac{3 \sin \alpha}{1 + \cos \alpha + 4\,(1 - \cos \alpha)} = \frac{3 \sin \alpha}{5 - 3 \cos \alpha}.$

126. $\sin (36° + A) - \sin (36° - A) = 2 \cos 36° \sin A = \frac{\sqrt{5}+1}{2} \sin A;$

$\sin (72° - A) - \sin (72° + A) = 2 \cos 72° \sin (-A) = -\frac{\sqrt{5}-1}{2} \sin A.$

By addition we obtain the required result.

127. $\tan \theta = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} = \frac{-\frac{2}{3}}{\pm \sqrt{\frac{5}{9}}} = \pm \frac{2}{\sqrt{5}}.$

The boundary line of θ is in the 3rd or 4th quadrant, hence the tangent is positive in one case and negative in the other.

128. (1) First side $= \sin 2A + \sin \left(\frac{\pi}{2} - 2B\right)$

$= 2 \sin \left(\frac{\pi}{4} + A - B\right) \cos \left(\frac{\pi}{4} - A - B\right).$

(2) First side $= 2 \sin \frac{\theta - \phi}{2} \cos \frac{\theta + \phi}{2} \,.\, 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2}$

$= 2 \sin (\theta - \phi) \cos^2 \frac{\theta + \phi}{2}.$

129. Since $a : b : c = \sin A : \sin B : \sin C$, we have

$$(a+b+c)(a+b-c)=3ab,$$

or
$$(a+b)^2-c^2=3ab,$$

that is,
$$\frac{a^2+b^2-c^2}{2ab}=\frac{1}{2};\ \therefore\ \cos C=\frac{1}{2},\ \text{and}\ C=60°.$$

130. Let $a^x=b,\quad a^y=d,\quad c^p=d,\quad c^q=b,$

then we have to prove $px=qy$.

Now
$$a^x=c^q,\ \text{and}\ a^y=c^p;$$
$$\therefore\ a^{px}=c^{pq}=a^{qy};\ \text{that is}\ px=qy.$$

131.
$$\log 20{\cdot}01=1{\cdot}3012471$$
$$\log 20{\cdot}00=1{\cdot}3010300$$
$$\text{diff. for } {\cdot}01=\overline{\quad 2171}$$

$$\therefore\ \text{diff. for } {\cdot}0075=\frac{3}{4}\times 2171=1628;$$

$$\therefore\ \log 20{\cdot}0075=1{\cdot}3011928.$$

132. Let AD be the median from B; then

$$AB^2+BC^2=2(AD^2+BD^2);$$

that is,
$$49+81=2x^2+32;$$

whence
$$x=7.$$

133. We have
$$\frac{1+\sin A}{\cos A}=2,$$

$$\therefore\ (1+\sin A)^2=4(1-\sin^2 A),$$

rejecting the factor $1+\sin A$, which gives an inadmissible value, we have

$$1+\sin A=4(1-\sin A);$$

whence
$$\sin A=\frac{3}{5}.$$

134.
$$\text{First side}=\frac{4(1-\cos 2A)-(1-\cos 4A)}{4(1+\cos 2A)-(1-\cos 4A)}$$
$$=\frac{8\sin^2 A-2\sin^2 2A}{8\cos^2 A-2\sin^2 2A}$$
$$=\frac{\sin^2 A(1-\cos^2 A)}{\cos^2 A(1-\sin^2 A)}=\tan^4 A.$$

135. First side $= \dfrac{3\sin A - 4\sin^3 A + 4\cos^3 A - 3\cos A}{3\sin A - 4\sin^3 A - (4\cos^3 A - 3\cos A)}$

$= \dfrac{3(\sin A - \cos A) - 4(\sin^3 A - \cos^3 A)}{3(\sin A + \cos A) - 4(\sin^3 A + \cos^3 A)}$

$= \dfrac{\sin A - \cos A}{\sin A + \cos A} \cdot \dfrac{3 - 4(\sin^2 A + \cos^2 A + \sin A \cos A)}{3 - 4(\sin^2 A + \cos^2 A - \sin A \cos A)}$

$= \dfrac{\tan A - 1}{\tan A + 1} \cdot \dfrac{3 - 4(1 + \sin A \cos A)}{3 - 4(1 - \sin A \cos A)}$

$= \dfrac{\tan A - 1}{1 + \tan A} \cdot \dfrac{1 + 2\sin 2A}{1 - 2\sin 2A} = \dfrac{1 + 2\sin 2A}{1 - 2\sin 2A} \tan(A - 45°).$

136. Put $\dfrac{x}{\cos A} = \dfrac{y}{\cos B} = k$; then

First side $= k\cos A \tan A + k\cos B \tan B$

$= k(\sin A + \sin B)$

$= 2k \sin \dfrac{A+B}{2} \cos \dfrac{A-B}{2}$

$= 2k \cos \dfrac{A+B}{2} \cos \dfrac{A-B}{2} \tan \dfrac{A+B}{2}$

$= k(\cos A + \cos B) \tan \dfrac{A+B}{2}$

$= (x+y) \tan \dfrac{A+B}{2}.$

137.

$\log 7 = \log 24{\cdot}5 - \log 3{\cdot}5 = {\cdot}845098;$

$\log 5 = \log 35 - \log 7 = {\cdot}69897;$

$\log 13 = \log 3{\cdot}25 - \log {\cdot}25$

$= \log 3{\cdot}25 - (2\log 5 - 2) = 1{\cdot}113943.$

138. Here $\tan A = \dfrac{a}{b} = \dfrac{384}{330} = \dfrac{128}{110}.$

$\log 128 = 7\log 2$	$= 2{\cdot}1072100$
$\log 110$	$= 2{\cdot}0413927$
$\log \tan A$	$= {\cdot}0658173$
$\log \tan 49° 19'$	$= {\cdot}0656886$
diff.	1287

$\therefore$ prop^{l} increase $= \dfrac{1287}{2555} \times 60'' = 30'';$

$\therefore\ A = 49° 19' 30''; \quad B = 40° 40' 30''.$

139. First side $= \dfrac{2\sin\dfrac{\theta+\alpha}{2}\sin\dfrac{\theta-\alpha}{2}}{2\cos\dfrac{\theta+\alpha}{2}\cos\dfrac{\theta-\alpha}{2}} = \dfrac{\cos\alpha-\cos\theta}{\cos\alpha+\cos\theta}$

$$= \frac{\cos\alpha-\cos\alpha\cos\beta}{\cos\alpha+\cos\alpha\cos\beta} = \frac{1-\cos\beta}{1+\cos\beta}$$

$$= \tan^2\frac{\beta}{2}.$$

140. We have

$$\frac{\sin\theta}{\cos\theta-\sin\phi} - \frac{\sin\theta}{\cos\theta+\sin\phi} = \frac{2\sin\theta\sin\phi}{\cos^2\theta-\sin^2\phi}$$

$$= \frac{\sin\phi\{\cos\phi+\sin\theta-(\cos\phi-\sin\theta)\}}{1-\sin^2\theta-(1-\cos^2\phi)}$$

$$= \frac{\sin\phi\{\cos\phi+\sin\theta-(\cos\phi-\sin\theta)\}}{(\cos\phi+\sin\theta)(\cos\phi-\sin\theta)}$$

$$= \frac{\sin\phi}{\cos\phi-\sin\theta} - \frac{\sin\phi}{\cos\phi+\sin\theta}.$$

141. We have $\dfrac{c^2(a+b)^2s(s-b)}{ac} = \dfrac{b^2(a+c)^2s(s-c)}{ab}$;

$$c(a^2+b^2+2ab)(s-b) = b(a^2+c^2+2ac)(s-c);$$

that is, $$a^2s(c-b) - bcs(c-b) + bc(c^2-b^2) + 2abc(c-b) = 0,$$

or $$(c-b)\{a^2s - bc(s-c-b) + 2abc\} = 0,$$

or $$(c-b)\{a^2s - bc(a-s) + 2abc\} = 0,$$

or $$(c-b)\{a^2s + bcs + 3abc\} = 0;$$

therefore $b-c=0$, since the other factor evidently cannot be zero.

142. (1) $\cot A + \operatorname{cosec} A = \dfrac{1+\cos A}{\sin A} = \cot\dfrac{A}{2}$; [XI. *d.* Ex. 11]

$$\tan A + \sec A = \frac{1+\sin A}{\cos A} = \frac{1-\cos\left(\frac{\pi}{2}+A\right)}{\sin\left(\frac{\pi}{2}+A\right)}$$

$$= \tan\left(\frac{\pi}{4}+\frac{A}{2}\right);$$

therefore by division the required result is obtained.

(2) Since $\sin 3A = 3\sin A - 4\sin^3 A$, we have

$$\text{First side} = \frac{1}{4}\{(3\sin A - \sin 3A) + \ldots + \ldots\}$$

$$= \frac{3}{4}\{\sin A + \sin(120° + A) + \sin(240° + A)\}$$

$$- \frac{1}{4}\{\sin 3A + \sin(360° + 3A) + \sin(720° + 3A)\}$$

$$= -\frac{3}{4}\sin 3A. \qquad \text{[See solution of XII. c. 25.]}$$

143. Let $x = \sqrt[5]{18 \times \cdot0015}$,

then $x = \sqrt[5]{\cdot027} = (\cdot3)^{\frac{3}{5}}$,

$$\frac{3}{5}\log \cdot3 = \frac{3}{5}(\bar{1}\cdot4771213)$$

$$= \bar{1}\cdot6862728 \qquad \text{diff. for } \cdot00001 = 31;$$

$$\log \cdot48559 = \bar{1}\cdot6862697 \qquad \therefore \text{ prop}^{\text{l}} \text{ increase} = \frac{31}{90} \times \cdot00001$$

$$31 \qquad = \cdot000003;$$

$$\therefore\ x = \cdot485593.$$

144.
$$\sin B = \frac{b\sin A}{a} = \frac{394}{573}\cos 22° 4'.$$

$\log 394$	$= 2\cdot5954962$
$\log \cos 22° 4'$	$= \bar{1}\cdot9669614$
	$2\cdot5624576$
$\log 573$	$= 2\cdot7581546$
$\log \sin B$	$= \bar{1}\cdot8043030$
$\log \sin 39° 35'$	$= \bar{1}\cdot8042757$
diff.	273
diff. for $60''$	$= 1527;$

$$\therefore \text{ prop}^{\text{l}} \text{ increase} = \frac{273}{1527} \times 60'' = 10\cdot7'';$$

$$\therefore\ B = 39° 35' 11''; \text{ and } C = 28° 20' 49''.$$

145. $$\text{First side} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc}$$

$$= \frac{a^2 + b^2 + c^2}{2abc}.$$

146. $\log 119 = \log 7 + \log 17 = 2{\cdot}0755469,$

$$\log \frac{17}{7} = \log 17 - \log 7 = {\cdot}3853509,$$

$$\log \frac{289}{343} = \log 17^2 - \log 7^3$$

$$= 2\log 17 - 3\log 7 = \bar{1}{\cdot}9256038.$$

147. We have $\cos\theta = \dfrac{\sin A}{\sin B + \sin C}$

$$= \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}} = \frac{\cos\frac{B+C}{2}}{\cos\frac{B-C}{2}};$$

$$\therefore \tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{\cos\frac{B-C}{2} - \cos\frac{B+C}{2}}{\cos\frac{B-C}{2} + \cos\frac{B+C}{2}}$$

$$= \frac{2\sin\frac{B}{2}\sin\frac{C}{2}}{2\cos\frac{B}{2}\cos\frac{C}{2}} = \tan\frac{B}{2}\tan\frac{C}{2}.$$

148. Let r be the radius of the circle, x, y the side of circumscribing equilateral triangle and hexagon respectively.

Then from the figure of Art. 215,

$$x = 2r\tan 60° = 2r\sqrt{3}; \quad y = 2r\tan 30 = \frac{2r}{\sqrt{3}};$$

whence $$xy = 4r^2 = (2r)^2.$$

149. From the equation $a^2 = b^2 + c^2 - 2bc\cos A$, we have on substitution and reduction

$$c^2 - 150\sqrt{2}\,.\,c + 10000 = 0;$$

$$\therefore (c - 100\sqrt{2})(c - 50\sqrt{2}) = 0;$$

$$\therefore c = 100\sqrt{2}, \text{ or } 50\sqrt{2}.$$

Again $$\sin B = \frac{b \sin A}{a} = \frac{150}{50\sqrt{10}} = \frac{3}{\sqrt{10}},$$

$$\log \sin B = \log 3 - \frac{1}{2}\log 10$$

$$= \bar{1}\cdot 9771213,$$

$$\log \sin 71^\circ 33' = \bar{1}\cdot 9770832$$

diff. $\qquad\qquad 381$

diff. for $60'' = 421$;

$\therefore$ prop^l increase $= \frac{381}{421} \times 60'' = 54''$;

$\therefore$ $B = 71^\circ 33' 54''$, or $180^\circ - (71^\circ 33' 54'')$.

150. The series may be written

$$(\operatorname{cosec} x - \operatorname{cosec} 3x) + (\operatorname{cosec} 3x - \operatorname{cosec} 3^2 x) + \ldots + (\operatorname{cosec} 3^{n-1} x - \operatorname{cosec} 3^n x),$$

which reduces to $\operatorname{cosec} x - \operatorname{cosec} 3^n x$.

153. First side $$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{3\cos\frac{3\theta}{2}}{\sin\frac{3\theta}{2}}$$

$$= \frac{\sin\frac{3\theta}{2}\cos\frac{\theta}{2} - 3\sin\frac{\theta}{2}\cos\frac{3\theta}{2}}{\sin\frac{\theta}{2}\sin\frac{3\theta}{2}}$$

$$= \frac{\sin\left(\frac{3\theta}{2} - \frac{\theta}{2}\right) - 2\sin\frac{\theta}{2}\cos\frac{3\theta}{2}}{\sin\frac{\theta}{2}\sin\frac{3\theta}{2}}$$

$$= \frac{\sin\theta - (\sin 2\theta - \sin\theta)}{\frac{1}{2}(\cos\theta - \cos 2\theta)}$$

$$= \frac{4\sin\theta - 2\sin 2\theta}{1 + \cos\theta - 2\cos^2\theta} = \frac{4\sin\theta(1-\cos\theta)}{(1+2\cos\theta)(1-\cos\theta)}$$

$$= \frac{4\sin\theta}{1+2\cos\theta}.$$

154. We have, by Art. 135, Ex. 2,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C;$$

$$\therefore \frac{3}{4}+\frac{5}{12}+\tan C=\frac{3}{4}\cdot\frac{5}{12}\cdot\tan C;$$

whence

$$\tan C=-\frac{56}{33}.$$

Also

$$\cos A=\frac{1}{\sqrt{1+\tan^2 A}}=\frac{1}{\sqrt{1+\frac{9}{16}}}=\frac{4}{5},$$

$$\cos B=\frac{1}{\sqrt{1+\tan^2 B}}=\frac{1}{\sqrt{1+\frac{25}{144}}}=\frac{12}{13},$$

$$\cos C=\frac{1}{\sqrt{1+\tan^2 C}}=\frac{1}{\sqrt{1+\frac{3136}{1089}}}=-\frac{33}{65},$$

the negative sign of the radical being taken in the third case since C is an obtuse angle.

Again

$$\tan C=-\frac{56}{33};\quad\therefore\quad \tan(180^\circ-C)=\frac{56}{33},$$

$$\begin{aligned}\log 56&=1{\cdot}7481880\\ \log 33&=1{\cdot}5185139\\ \log\tan(180^\circ-C)&=\cdot2296741\\ \log\tan 59^\circ 29'&=\cdot2295627\\ \text{diff.}&\quad\overline{1114}\end{aligned}$$

$$\therefore \text{ prop}^{\text{l}}\text{ increase}=\frac{1114}{2888}\times 60''=23'';$$

$$\therefore 180^\circ-C=59^\circ 29' 23'';$$

that is,

$$C=120^\circ 30' 37''.$$

155. We have

$$\frac{\sin(A-B)}{\sin(A+B)}=\frac{a^2-b^2}{a^2+b^2}$$

$$=\frac{\sin^2 A-\sin^2 B}{\sin^2 A+\sin^2 B};$$

$$\therefore \frac{\sin(A-B)}{\sin C}=\frac{\sin(A-B)\sin(A+B)}{\sin^2 A+\sin^2 B};$$

$\therefore$ either

$$\sin(A-B)=0 \quad\ldots\ldots (1),$$

or

$$\sin^2 C=\sin^2 A+\sin^2 B \quad\ldots\ldots (2).$$

If (1) is true, $A=B$; if (2) is true, we have $c^2=a^2+b^2$.

156. The first side $= \frac{2\Delta}{s} \cdot \frac{abc}{\Delta} \left\{ \frac{s(s-a)}{bc} + \ldots + \ldots \right\}$

$= 2\{a(s-a) + b(s-b) + c(s-c)\}$

$= (a+b+c)^2 - 2a^2 - 2b^2 - 2c^2$

$= 2bc + 2ca + 2ab - a^2 - b^2 - c^2.$

157. We have

$$\frac{\cos C}{\sin B \cos A} - \frac{\cos B}{\sin C \cos A} = \frac{\cos (A+C)}{\sin C \cos A} - \frac{\cos (A+B)}{\sin B \cos A}$$

$$= \cot C - \tan A - (\cot B - \tan A)$$

$$= \cot C - \cot B.$$

158. $\log 3 = \log 18 - \log 6$; $\log 2 = \log 6 - \log 3$;

$\log 11 = \log 44 - \log 4 = \log 44 - 2\log 2.$

159. (1) We have

$$\tan (60° + A) \tan (60° - A)$$

$$= \frac{2\sin (60°+A)\sin (60°-A)}{2\cos (60°+A)\cos (60°-A)} = \frac{\cos 2A - \cos 120°}{\cos 2A + \cos 120°} = \frac{2\cos 2A + 1}{2\cos 2A - 1};$$

$\therefore$ second side $= 2\sin A \cos 2A + \sin A$

$= (\sin 3A - \sin A) + \sin A = \sin 3A.$

(2) First side $= 2\sin (A-B)\cos (A+B)\frac{\sin (A+B)}{\cos (A+B)}$

$= 2\sin (A+B)\sin (A-B)$

$= 2\sin^2 A - \sin^2 B.$ [Art. 114.]

160. $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{113 \times 101}{296 \times 82}}$

$\log 296 = 2{\cdot}4712917$	$\log 113 = 2{\cdot}0530784$
$\log 82 = 1{\cdot}9138139$	$\log 101 = 2{\cdot}0043214$
$4{\cdot}3851056$	$4{\cdot}0573998$
	$4{\cdot}3851056$
	$2 \mid \overline{1}{\cdot}6722942$
$\log \tan \frac{C}{2} =$	$\overline{1}{\cdot}8361471$
$\log \tan 34° 26' =$	$\overline{1}{\cdot}8360513$
diff.	958

$$\text{prop}^{\text{l}}. \text{ increase} = \frac{958}{2708} \times 60'' = 21'';$$

$\therefore C = 68° 52' 42''.$

161. Let a be a side of the octagon, r the radius of the circle, then $8a = 2\pi r$; and we have

$$\frac{\text{area of circle}}{\text{area of octagon}} = \frac{\pi r^2}{2a^2 \cot \frac{\pi}{8}} = \frac{8\pi r^2}{\pi^2 r^2} \tan \frac{\pi}{8}$$

$$= \frac{8(\sqrt{2}-1)}{\pi} = \frac{\cdot 414 \times 8}{3 \cdot 1416} = \frac{4140}{3927} = \frac{1380}{1309}.$$

162. We have $2b = a + c$, or $a = 2b - c$;

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2b-c)^2}{2bc} = \frac{4bc - 3b^2}{2bc} = \frac{4c - 3b}{2c}.$$

163. We have $a(\sin\theta \cos\alpha + \cos\theta \sin\alpha) = b(\sin\theta \cos\beta + \cos\theta \sin\beta)$;

$$\therefore \sin\theta(a\cos\alpha - b\cos\beta) = \cos\theta(b\sin\beta - a\sin\alpha);$$

that is,
$$\cot\theta = \frac{a\cos\alpha - b\cos\beta}{b\sin\beta - a\sin\alpha}.$$

164. Let a, b, A be the given parts; then $R = \frac{a}{2\sin A}$, which is the same for each triangle.

165. Let NS be a horizontal line pointing North and South. Then if K be the position of the kite, and KD is the vertical from K, we have S, B, D, A, N in a straight line, and $BA = c$. Also $KA = \frac{c\sin\beta}{\sin BKA} = \frac{c\sin\beta}{\sin(\alpha+\beta)}$.

And $KD = KA\sin\alpha = \frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

166.
$$\cos\alpha + \cos\beta + \cos\gamma + 1 = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2}$$

$$= 2\cos\left(\pi - \frac{\gamma}{2}\right)\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2}$$

$$= -2\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2}$$

$$= 2\cos\frac{\gamma}{2}\left\{\cos\frac{\gamma}{2} - \cos\frac{\alpha-\beta}{2}\right\}$$

$$= 2\cos\frac{\gamma}{2}\left\{\cos\left(\pi - \frac{\alpha+\beta}{2}\right) - \cos\frac{\alpha-\beta}{2}\right\}$$

$$= -2\cos\frac{\gamma}{2}\left\{\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2}\right\}$$

$$= -4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

167. First side $= 2\cos 9A\cos A + 6\cos 3A\cos A$

$= 2\cos A\{\cos 9A + 3\cos 3A\}$

$= 2\cos A\{(4\cos^3 3A - 3\cos 3A) + 3\cos 3A\}$

$= 8\cos A\cos^3 3A.$

168. $r_1 + r_2 = \dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-b} = \dfrac{\Delta(s-a+s-b)}{(s-a)(s-b)} = c\cdot\sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}.$

Also $r_2r_3 + r_3r_1 + r_1r_2 = s^2$ [XVIII. *a*. Ex. 24];

$\therefore$ First side $= \dfrac{abc}{4s^2}\sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}\cdot\dfrac{s(s-a)}{(s-b)(s-c)}\cdot\dfrac{s(s-b)}{(s-c)(s-a)}}$

$= \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \dfrac{abc}{4\Delta} = R.$

169. $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = -\dfrac{2\tan^2\phi}{2(1+\tan^2\phi)}$

$= -\sin^2\phi.$

170. First side $= \dfrac{\sin A\sin(60°+A)\sin(120°+A)}{\cos A\cos(60°+A)\cos(120°+A)}$

$= \dfrac{\sin A}{\cos A}\cdot\dfrac{\cos 60° - \cos(180°+2A)}{\cos 60° + \cos(180°+2A)}$

$= \dfrac{\sin A}{\cos A}\cdot\dfrac{1+2\cos 2A}{1-2\cos 2A}$

$= \dfrac{\sin A + 2\sin A\cos 2A}{\cos A - 2\cos A\cos 2A}$

$= \dfrac{\sin A + (\sin 3A - \sin A)}{\cos A - (\cos 3A + \cos A)}$

$= -\dfrac{\sin 3A}{\cos 3A} = -\tan 3A.$

171. We have $A - B = B$; therefore $\sin(A-B) = \sin B$;

also $\sin(A+B) = \sin(180° - C) = \sin C$;

$\therefore \sin(A+B)\sin(A-B) = \sin C\sin B$;

that is, $\sin^2 A - \sin^2 B = \sin B\sin C$;

or $a^2 - b^2 = bc.$

172. If a, b be the sides of the triangle and square respectively, and R the radius of the circle, it is easy to shew that

$$a = R\sqrt{3}\,;\quad b = 2R\cos 45^\circ = R\sqrt{2}.$$

173. We have $\dfrac{c+b}{c-b}\tan\dfrac{A}{2} = \dfrac{\sin C+\sin B}{\sin C-\sin B}\tan\dfrac{A}{2}$

$$= \frac{\sin(A+B)+\sin B}{\sin(A+B)-\sin B}\tan\frac{A}{2}$$

$$= \frac{2\sin\left(\frac{A}{2}+B\right)\cos\frac{A}{2}}{2\cos\left(\frac{A}{2}+B\right)\sin\frac{A}{2}}\tan\frac{A}{2}$$

$$= \tan\left(\frac{A}{2}+B\right).$$

$$\tan\left(\frac{A}{2}+B\right) = \frac{7b+3b}{7b-3b}\tan\frac{A}{2} = \frac{10}{4}\tan\frac{A}{2},$$

$$\log 10 = 1$$
$$\log 4 = \cdot 60206$$
$$\cdot 3979400$$
$$\log\tan 3^\circ 18' 42'' = \bar{2}\cdot 7624069$$
$$\log\tan\left(\frac{A}{2}+B\right) = \bar{1}\cdot 1603469$$
$$\log\tan 8^\circ 13' 50'' = \bar{1}\cdot 1603083$$
$$386$$

$$\text{prop}^{\text{l}}\text{ increase} = \frac{386}{1486}\times 60'' = 2\cdot 6''.$$

$$\therefore\ \frac{A}{2}+B = 8^\circ 13' 53'',\ \text{and}\ \frac{A}{2} = 3^\circ 18' 42'';$$

$$\therefore\ B = 4^\circ 55' 11'',$$
$$C = 168^\circ 27' 25''.$$

174. By a well-known geometrical property, we have

$$AC^2+AB^2 = 2AD^2+2DB^2.$$

$$\therefore\ AC^2-AB^2 = 2(AD^2+DB^2-AB^2)$$
$$= 4AD\,.\,DB\cos ADB$$
$$= 4AD\,.\,DB\sin ADB\cot ADB$$
$$= (AD\,.\,DB\sin ADB)\,4\cot ADB$$
$$= 4\Delta\cot ADB,$$

for $AD\,.\,DB\sin ADB = 2$ (area of triangle ADB) $= \Delta$.

175. $\tan 40^\circ \tan 80^\circ = \dfrac{2\sin 40^\circ \sin 80^\circ}{2\cos 40^\circ \cos 80^\circ}$

$$= \frac{\cos 40^\circ - \cos 120^\circ}{\cos 40^\circ + \cos 120^\circ}$$

$$= \frac{2\cos 40^\circ + 1}{2\cos 40^\circ - 1};$$

$$\therefore \tan 20^\circ \tan 40^\circ \tan 80^\circ = \frac{2\sin 20^\circ \cos 40^\circ + \sin 20^\circ}{2\cos 20^\circ \cos 40^\circ - \cos 20^\circ}$$

$$= \frac{\sin 60^\circ}{\cos 60^\circ} = \tan 60^\circ.$$

176. See Art. 144.

177. $2R = \dfrac{abc}{2\Delta}; \quad 2r = \dfrac{2\Delta}{s};$

$$\therefore 2R \,.\, 2r = \frac{abc}{s} = \frac{2abc}{a+b+c}.$$

178. $\sin B = \dfrac{b}{a}\sin A = \dfrac{4\sqrt{3}}{7}$

$\frac{1}{2}\log 3 = \cdot 2385606$

$2\log 2 = \cdot 6020600$

$\cdot 8406206$

$\log 7 = \cdot 8450980$

$\log \sin B = \bar{1}\cdot 9955226$

$\log \sin 81^\circ 47' = \bar{1}\cdot 9955188$

38

prop^{l}. increase $= \dfrac{38}{183} \times 60'' = 12\cdot 4''.$

$\therefore B = 81^\circ 47' 12''$; but since $a < b$ there is another value of B supplementary to this, viz. $98^\circ 12' 48''$.

$$\therefore C = 68^\circ 12' 48'', \text{ or } 51^\circ 47' 12''.$$

To find c, we have $c^2 - 2b\cos A \,.\, c + b^2 - a^2 = 0,$ [Art. 150]

that is $$c^2 - 24c + 143 = 0,$$

$$(c-13)(c-11) = 0;$$

$$\therefore c = 13, \text{ or } 11.$$

179. $\tan^2\left(\frac{\pi}{4}+\beta\right)=\frac{1+\sin 2\beta}{1-\sin 2\beta}$ [XI. *f.* Ex. 15]

$$=\frac{1+\sin 2a\sin 2a'+\sin 2a+\sin 2a'}{1+\sin 2a\sin 2a'-\sin 2a-\sin 2a'}$$

$$=\frac{(1+\sin 2a)(1+\sin 2a')}{(1-\sin 2a)(1-\sin 2a')}$$

$$=\tan^2\left(\frac{\pi}{4}+a\right)\tan^2\left(\frac{\pi}{4}+a'\right).$$

180. With the figure of Art. 199, let

$$PC=x,\ \beta=28°,\ a=16°,\ a=16071 \text{ feet.}$$

Then $$x=\frac{16071\sin 28°\sin 16°}{\sin 12°}.$$

$$\log 16071=4\cdot2060$$
$$\log\sin 28°=\bar{1}\cdot6716$$
$$\log\sin 16°=\bar{1}\cdot4403$$
$$\overline{3\cdot3179}$$
$$\log\sin 12°=\bar{1}\cdot3179$$
$$\log x=\overline{4\cdot0000}$$
$$\therefore\ x=10000 \text{ feet.}$$

182. We have $2\cos\frac{A+C}{2}\cos\frac{A-C}{2}=2\sin(A+C)$

$$=4\sin\frac{A+C}{2}\cos\frac{A+C}{2};$$

$\therefore$ either $$\cos\frac{A+C}{2}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or $$\cos\frac{A-C}{2}=2\sin\frac{A+C}{2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) $\frac{A+C}{2}=(2n+1)\frac{\pi}{2}$, and from (2) by expanding each side and dividing throughout by $\cos\frac{A}{2}\cos\frac{C}{2}$ we obtain the other result.

183. First side $=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}-2\sin\frac{C}{2}\cos\frac{C}{2}$

$$=2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\cos\frac{C}{2}\right\}$$

$$= 2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{\pi-(A+B)}{2}\right\}$$

$$= 2\sin\frac{C}{2}\cdot 2\sin\left(\frac{\pi}{4}-\frac{B}{2}\right)\sin\left(\frac{\pi}{4}-\frac{A}{2}\right)$$

$$= 4\sin\frac{C}{2}\sin\left(45°-\frac{A}{2}\right)\sin\left(45°-\frac{B}{2}\right).$$

184. $\dfrac{bc}{r_1} = \dfrac{bc(s-a)}{\Delta} = \dfrac{4R(s-a)}{a} = 2R\left(\dfrac{b+c-a}{a}\right)$;

$$\therefore \text{ first side} = 2R\left(\frac{b+c-a}{a} + \frac{c+a-b}{b} + \frac{a+b-c}{c}\right)$$

$$= 2R\left(\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}-3\right).$$

185. Since the points A, B, E, C are concyclic, $\angle BED = \angle C$; also

$$\angle EBD = \angle EAC = \frac{A}{2};$$

$\therefore$ from $\triangle BDE$, $$BD = \frac{DE\sin C}{\sin\frac{A}{2}},$$

from $\triangle DEC$, $$DC = \frac{DE\sin B}{\sin\frac{A}{2}};$$

$$\therefore \text{ by addition } a = \frac{DE}{\sin\frac{A}{2}}(\sin B + \sin C);$$

$$\therefore a^2 = \frac{DE}{\sin\frac{A}{2}}(a\sin B + a\sin C)$$

$$= \frac{DE(b+c)\sin A}{\sin\frac{A}{2}};$$

$$\therefore DE = \frac{a^2\sec\frac{A}{2}}{2(b+c)}.$$

186. $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{5875{\cdot}5\times1785{\cdot}5}{3850\times3811}}$

$\log 5875{\cdot}5=3{\cdot}7690448$ $\qquad$ $\log 3850=3{\cdot}5854607$

$\log 1785{\cdot}5=3{\cdot}2517599$ $\qquad$ $\log 3811=3{\cdot}5810389$

$7{\cdot}0208047$ $\qquad$ $7{\cdot}1664996$

$7{\cdot}1664996$

$2\,\overline{1}{\cdot}8543051$

$\log\cos\frac{A}{2}=\overline{1}{\cdot}9271525$

$\log\cos 32^\circ 16'=\overline{1}{\cdot}9271509$

16 $\qquad$ prop^{l}. *decrease* $=\frac{16}{797}\times 60''$

$=1{\cdot}2''$;

$$\therefore\ \frac{A}{2}=32^\circ\,15'\,58{\cdot}8'',\ \text{or}\ A=64^\circ\,31'\,58''.$$

188. See Art. 117.

190. (1) $\frac{b^2+c^2-a^2}{a^2+c^2-b^2}=\frac{2bc\cos A}{2ca\cos B}=\frac{b\cos A}{a\cos B}$

$$=\frac{\sin B\cos A}{\sin A\cos B}=\frac{\tan B}{\tan A}.$$

(2) $\frac{2\sin^2 A}{a^2}=\frac{2\sin^2 B}{b^2}$;

$$\therefore\ \frac{1-\cos 2A}{a^2}=\frac{1-\cos 2B}{b^2},$$

that is,
$$\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2}=\frac{1}{a^2}-\frac{1}{b^2}.$$

191. $\Delta=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{110\,.\,33\,.\,42\,.\,35}$

$=\sqrt{11^2\,.\,7^2\,.\,3^2\,.\,2^2\,.\,5^2}=11\,.\,7\,.\,3\,.\,2\,.\,5$

$=2310$ sq. ft.

$$r_1=\frac{\Delta}{s-a}=\frac{2310}{33}=70\text{ ft.}$$

$$r_2=\frac{\Delta}{s-b}=\frac{2310}{42}=55\text{ ft.}$$

$$r_3=\frac{\Delta}{s-c}=\frac{2310}{35}=66\text{ ft.}$$

192. Draw DK, DK' perpendicular to AB, AC respectively; then

$$DK\,.\,AB+DK'\,.\,AC=2\Delta;$$

that is,
$$c\,.\,AD\sin\frac{A}{2}+b\,.\,AD\sin\frac{A}{2}=bc\sin A;$$

$$\therefore\ AD\,(b+c)=2bc\cos\frac{A}{2}.$$

193. (1) $\sin 5\theta - \sin 3\theta = \sqrt{2}\sin\theta.$

$$\therefore\ 2\sin\theta\cos 4\theta = \sqrt{2}\sin\theta;$$

that is, $$\sin\theta = 0, \text{ or } \cos 4\theta = \frac{1}{\sqrt{2}}.$$

$$\therefore\ \theta = n\pi, \text{ or } 4\theta = 2n\pi \pm \frac{\pi}{4}.$$

(2) $$\cot\theta + \cot\left(\frac{\pi}{4} + \theta\right) = 2;$$

$$\therefore\ \cot\theta + \frac{\cot\theta - 1}{\cot\theta + 1} = 2;$$

$$\cot^2\theta + 2\cot\theta - 1 = 2\cot\theta + 2;$$

$$\cot^2\theta = 3;$$

that is, $$\cot\theta = \pm\sqrt{3}, \text{ and } \theta = n\pi \pm \frac{\pi}{6}.$$

194. The given relation easily reduces to $\cos 2\alpha = \sin 2\beta$, one solution of which is $2\alpha = \frac{\pi}{2} - 2\beta$.

195. We have $$\tan(\alpha+\theta)\tan(\alpha-\theta) = \tan^2\beta;$$

$$\therefore\ \frac{\tan^2\alpha - \tan^2\theta}{1 - \tan^2\alpha\tan^2\theta} = \tan^2\beta;$$

whence $$\tan^2\theta(1 - \tan^2\alpha\tan^2\beta) = \tan^2\alpha - \tan^2\beta;$$

$$\therefore\ \tan^2\theta = \frac{(\tan\alpha + \tan\beta)(\tan\alpha - \tan\beta)}{(1 - \tan\alpha\tan\beta)(1 + \tan\alpha\tan\beta)}$$

$$= \tan(\alpha+\beta)\tan(\alpha-\beta).$$

196. (1) We have $$p_1 = \frac{2\Delta}{a};$$

$\therefore$ second side $$= \frac{a^3b^3c^3}{8\Delta^3} = 8\left(\frac{abc}{4\Delta}\right)^3 = 8R^3.$$

(2) Second side $$= \frac{a^2 + b^2 - 2ab\cos C}{4\Delta^2}$$

$$= \frac{c^2}{4\Delta^2} = \left(\frac{c}{2\Delta}\right)^2 = \frac{1}{p_3^2}.$$

197. By Art. 215 we have

$$\text{perimeter} = 30r \tan \frac{\pi}{15}, \text{ where } \pi r^2 = 1386.$$

Now $$r^2 = \frac{7}{22} \times 1386 = 7 \times 63; \quad \therefore r = 21;$$

$$\therefore \text{ perimeter} = 30 \times 21 \tan 12°$$
$$= 630 \times \cdot 213$$
$$= 134 \cdot 19 \text{ ft.}$$

198. Let A, B represent the foot of the pole in the two positions; C, S the top of the pole on the coping and sill respectively; also let W be the foot of the wall.

Then $$x + SW = AC \sin \alpha,$$

but $$SW = BS \sin \beta = AC \sin \beta;$$

$$\therefore x = AC (\sin \alpha - \sin \beta).$$

Similarly $$a = AC (\cos \beta - \cos \alpha);$$

$$\therefore \frac{x}{a} = \frac{2 \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}}{2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}} = \cot \frac{\alpha+\beta}{2}.$$

199. See Examples XVIII. *a.* 18. Each of the three expressions will be found to be equal to r.

200. (1) Let $\sin^{-1} \frac{2}{7} = \theta$; then $\cos 2\theta = 1 - 2 \sin^2 \theta = \frac{41}{49}$;

$$\therefore \cos^{-1} \frac{41}{49} = 2\theta = 2 \sin^{-1} \frac{2}{7}.$$

(2) $$3 \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{3 \times \frac{1}{4} - \left(\frac{1}{4}\right)^3}{1 - 3\left(\frac{1}{4}\right)^2}$$

$$= \tan^{-1} \frac{3 \times 16 - 1}{64 - 12} = \tan^{-1} \frac{47}{52}.$$

201. (1) As in XI. *f.* Ex. 14 we may prove that

$$\tan A + \sec A = \tan \left(45° + \frac{A}{2}\right).$$

Also $$\cot A + \operatorname{cosec} A = \frac{1 + \cos A}{\sin A} = \cot \frac{A}{2};$$

$$\therefore (\tan A + \sec A) \cot \frac{A}{2} = (\cot A + \operatorname{cosec} A) \tan \left(45° + \frac{A}{2}\right).$$

(2) First side

$= 2\cos(A+B)\cos(A-B) - 2\cos(A+B)\{\cos(A-B) - \cos(90° - \overline{A+B})\}$

$= 2\cos(A+B)\sin(A+B) = \sin 2(A+B).$

202. By Art. 214, $\frac{\text{perimeter of fig.}}{\text{diameter of circle}} = 7\sin 25\tfrac{1}{7}°,$

$\log 7\sin 25\tfrac{1}{7}° = \cdot 8450980 + \bar{1}\cdot 6373733$

$= \cdot 4824713,$ which is greater than $\log 3$.

203. We have $c^2 = a^2 + b^2 - 2ab\cos C$

$= (a^2+b^2)\left(\cos^2\frac{C}{2} + \sin^2\frac{C}{2}\right) - 2ab\left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right)$

$= (a+b)^2\sin^2\frac{C}{2} + (a-b)^2\cos^2\frac{C}{2}$

$= (a+b)^2\sin^2\frac{C}{2}\left\{1 + \left(\frac{a-b}{a+b}\right)^2\cot^2\frac{C}{2}\right\}$

$= (a+b)^2\sin^2\frac{C}{2}(1+\tan^2\phi)$

$= (a+b)^2\sin^2\frac{C}{2}\sec^2\phi;$

$\therefore\ c = (a+b)\sin\frac{C}{2}\sec\phi.$

204. We have $\tan\phi = \frac{237-158}{237+158}\cot 33°\,20'$

$= \frac{2}{10}\cot 33°\,20';$

$\therefore \log\tan\phi = \log 2 - 1 + \log\cot 33°\,20'$

$= \bar{1}\cdot 30103 + \cdot 18197$

$= \bar{1}\cdot 48300$

$\log\tan 16°\,54' = \bar{1}\cdot 48262$

diff. $\quad 38$

prop^l increase $= \frac{38}{46}\times 60''$

$= 50'';$

$\therefore\ \phi = 16°\,54'\,50'',$

$\log\sec 16°\,55' = \cdot 01921$

$\log\sec 16°\,54' = \cdot 01917$

diff. for $60'' \quad 4$

prop^l increase for $50'' = \frac{4}{60}\times 50''$

$= 3'';$

$\therefore \log\sec\phi = \cdot 01920.$

Now $$c=(a+b)\sin\frac{C}{2}\sec\phi=395\sin\frac{C}{2}\sec\phi,$$

$$\log 395=\log 79+1-\log 2$$
$$=2{\cdot}59660$$
$$\log\sin\frac{C}{2}=\bar{1}{\cdot}73998$$
$$\log\sec\phi=\ \ {\cdot}01920$$
$$\log c=\bar{2}{\cdot}35578$$
$$\therefore\ c=226{\cdot}87.$$

205. $$2\cos^2 2\theta=1+\cos 4\theta;$$
$$\therefore\ 2\cos 2\theta=\sqrt{2+2\cos 4\theta}.$$

Similarly $$2\cos\theta=\sqrt{2+2\cos 2\theta}=\sqrt{2+\sqrt{2+2\cos 4\theta}}.$$

206. Let $$\sin^{-1}\frac{3}{\sqrt{73}}=\alpha,\ \text{so that}\ \cos\alpha=\frac{8}{\sqrt{73}},$$

and let $$\cos^{-1}\frac{11}{\sqrt{146}}=\beta,\ \text{so that}\ \sin\beta=\frac{5}{\sqrt{146}}.$$

Then $$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$=\frac{3}{\sqrt{73}}\cdot\frac{11}{\sqrt{146}}+\frac{8}{\sqrt{73}}\cdot\frac{5}{\sqrt{146}}=\frac{73}{73\sqrt{2}}=\frac{1}{\sqrt{2}}$$
$$=\sin\frac{\pi}{4}=\sin\left(\frac{5\pi}{12}-\frac{\pi}{6}\right);$$
$$\therefore\ \alpha+\beta=\frac{5\pi}{12}-\frac{\pi}{6}=\frac{5\pi}{12}-\sin^{-1}\frac{1}{2}.$$

Again $$\tan^{-1}\frac{\dfrac{x-1}{x+1}+\dfrac{2x-1}{2x+1}}{1-\dfrac{x-1}{x+1}\cdot\dfrac{2x-1}{2x+1}}=\tan^{-1}\frac{23}{36};$$
$$\tan^{-1}\frac{4x^2-2}{6x}=\tan^{-1}\frac{23}{36};$$
$$\therefore\ 36(2x^2-1)=69x,$$

or $$24x^2-23x-12=0;$$
$$\therefore\ (3x-4)(8x+3)=0,$$

that is, $$x=\frac{4}{3},\ \text{or}\ -\frac{3}{8}.$$

207. We have $$x=\frac{2\Delta}{a},\ R=\frac{abc}{4\Delta};$$

$$\therefore\ x=\frac{1}{a}\cdot\frac{abc}{2R};\ \therefore\ \frac{bx}{c}=\frac{b^2}{2R},$$

that is, $$\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}=\frac{a^2+b^2+c^2}{2R}.$$

208. We have $$\cos(\alpha+\theta)=\cos\left\{\frac{\pi}{2}-(\alpha-\theta)\right\},$$

$$\therefore\ \alpha+\theta=2m\pi\pm\left\{\frac{\pi}{2}-(\alpha-\theta)\right\};$$

the upper sign gives $$2\alpha=2m\pi+\frac{\pi}{2},$$

and the lower sign gives $$2\theta=2m\pi-\frac{\pi}{2}.$$

209. With the notation of Art. 228,

$$SI^2=R^2-2Rr.$$

If a be the base of the triangle, $A=120°$,

$$B=C=30°;\ \therefore\ r=4R\sin 60°\sin 15°\sin 15°;$$

$$\therefore\ SI^2=R^2-8R^2\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2$$

$$=R^2(4-2\sqrt{3});$$

$$\therefore\ SI=R(\sqrt{3}-1)$$

$$=\frac{a}{2\sin A}(\sqrt{3}-1)=\frac{a(\sqrt{3}-1)}{\sqrt{3}},$$

$$\therefore\ SI:a=\sqrt{3}-1:\sqrt{3}.$$

210. $$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=1+\frac{r}{R}=1+\frac{3}{4},$$

$$\therefore\ 4(\cos A+\cos B+\cos C)=7.$$

211. Since $$\frac{\cos(\theta-\alpha)}{\sin(\theta+\alpha)}=\frac{1+m}{1-m},$$

dividendo and componendo, we have

$$\frac{\cos(\theta-\alpha)-\sin(\theta+\alpha)}{\cos(\theta-\alpha)+\sin(\theta+\alpha)}=m.$$

By expanding the sines and cosines we obtain

$$\frac{(\cos\theta-\sin\theta)(\cos\alpha-\sin\alpha)}{(\cos\theta+\sin\theta)(\cos\alpha+\sin\alpha)}=m,$$

or $$\frac{1-\tan\theta}{1+\tan\theta}=m\left(\frac{\cot\alpha+1}{\cot\alpha-1}\right).$$ [See XI. *b*. Ex. 6, 7.]

212. (1) $$\sin 5\theta-\sin 3\theta=\sqrt{2}\cos 4\theta;$$

$$2\cos 4\theta\sin\theta=\sqrt{2}\cos 4\theta;$$

$$\therefore\ \cos 4\theta=0;\ \text{whence } 4\theta=(2n+1)\frac{\pi}{2},$$

or $$\sin\theta=\frac{1}{\sqrt{2}};\ \text{whence } \theta=n\pi+(-1)^n\frac{\pi}{4}.$$

(2) $$1+\sin 2\theta=\frac{1+\tan\theta}{1-\tan\theta}.$$

$$1+\frac{2\tan\theta}{1+\tan^2\theta}=\frac{1+\tan\theta}{1-\tan\theta}.$$

$$(1+\tan\theta)^2(1-\tan\theta)=(1+\tan\theta)(1+\tan^2\theta);$$

$$\therefore\ 1+\tan\theta=0;\ \text{whence } \theta=n\pi+\frac{3\pi}{4},$$

or $$1-\tan^2\theta=1+\tan^2\theta;$$

$$\therefore\ \tan\theta=0;\ \text{whence } \theta=n\pi.$$

213. We have $$2\cos\frac{A+B}{2}\cos\frac{A-B}{2}=4\sin^2\frac{C}{2};$$

$$\therefore\ \cos\frac{A-B}{2}=2\sin\frac{C}{2},\ \text{or } 2\cos\frac{A-B}{2}\sin\frac{A+B}{2}=4\sin\frac{C}{2}\cos\frac{C}{2};$$

that is, $$\sin A+\sin B=2\sin C,\ \text{or } a+b=2c.$$

214. With the figure on p. 186, we have $\tan\beta=\frac{1}{9}$, $PA=80$ ft. $CA=100$ ft. Let $BP=x$ ft., then

$$\tan\theta=\frac{x+80}{100},\ \tan(\theta-\beta)=\frac{80}{100}=\frac{4}{5},$$

$$\therefore\ \frac{\tan\theta-\tan\beta}{1+\tan\theta\tan\beta}=\frac{4}{5},$$

$$\frac{\frac{x+80}{100}-\frac{1}{9}}{1+\frac{x+80}{900}}=\frac{4}{5};$$

$$\therefore\ 5(9x+720-100)=4(980+x);$$

$$45x+3100=3920+4x;$$

$$41x=820;\ \text{or } x=20.$$

215. (1) $\cot^{-1}7+\cot^{-1}8=\cot^{-1}\dfrac{7.8-1}{7+8}=\cot^{-1}\dfrac{55}{15}$,

$$\cot^{-1}3-\cot^{-1}18=\cot^{-1}\frac{3.18+1}{18-3}=\cot^{-1}\frac{55}{15}.$$

(2) $$4\tan^{-1}\frac{1}{5}=2\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{5^2}}=2\tan^{-1}\frac{5}{12}$$

$$=\tan^{-1}\frac{\frac{2\times5}{12}}{1-\frac{5^2}{12^2}}=\tan^{-1}\frac{120}{119}.$$

$$\therefore\ 4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\tan^{-1}\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}=\tan^{-1}\frac{120.239-119}{119.239+120}$$

$$=\tan^{-1}\frac{119.239+(239-119)}{119.239+120}=\tan^{-1}1=\frac{\pi}{4}.$$

216. Proceeding as in Art. 259, Ex. 2, we find that $\frac{A}{2}$ lies between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$; that is A lies between $(8n+3)\frac{\pi}{2}$ and $(8n+5)\frac{\pi}{2}$.

217. First side $$=\frac{1}{2}\left\{1-\cos\left(\frac{\pi}{4}+\theta\right)-1+\cos\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$=\sin\frac{\pi}{4}\sin\theta=\frac{1}{\sqrt{2}}\sin\theta.$$

218. From the two given relations we easily deduce

$$x=\frac{\sin\theta}{\sin(\theta+\phi)},\quad y=\frac{\sin\phi}{\sin(\theta+\phi)},$$

$$\therefore\ \sin\theta:\sin\phi=x:y.$$

219.
$$\tan^{-1}\frac{x+1+x-1}{1-(x^2-1)}=\tan^{-1}\frac{8}{31};$$

$$\therefore\ \frac{2x}{2-x^2}=\frac{8}{31},\ \text{or}\ 4x^2+31x-8=0;$$

$$\therefore\ (4x-1)(x+8)=0,\ \text{or}\ x=\frac{1}{4},\ \text{or}\ -8.$$

Again $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)$

$=1+\tan^2(\tan^{-1}2)+1+\cot^2(\cot^{-1}3)$

$=1+4+1+9=15.$

220. Let $5A=\alpha$, $5B=\beta$, $5C=\gamma$, then $\alpha+\beta+\gamma=5\pi$;

$\therefore \sin(\alpha+\beta)=\sin(\pi-\gamma)=\sin\gamma$; $\cos\gamma=-\cos(\alpha+\beta)$,

$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=2\sin(\alpha+\beta)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma$

$=2\sin\gamma\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}$

$=4\sin\alpha\sin\beta\sin\gamma.$

In the second case the sum of the three angles is $\frac{16\pi}{2^5}$, or $\frac{\pi}{2}$ and the result easily follows as in Art. 135, Ex. 2.

221. See solution to XVIII. *a*. Ex. 24.

222. We have $x=\frac{BD\sin 15^\circ}{\sin 50^\circ}$; $BD=\frac{100}{\cos 25^\circ}$;

$$\therefore x=\frac{100\cos 75^\circ}{\cos 40^\circ\cos 25^\circ};$$

$\therefore \log x=2+\log\cos 75^\circ-(\log\cos 40^\circ+\log\cos 25^\circ)$

$=1\cdot4129962-(\bar{1}\cdot8415297)$

$=1\cdot5714665$

$\log 37\cdot279=1\cdot5714643$

diff. 22

prop^{l}. increase $=\frac{22}{116}\times\cdot001$

$=\cdot00019;$

$\therefore x=37\cdot27919.$

223. $\{\sec\theta+\operatorname{cosec}\theta(1+\sec\theta)\}^2=\left(\frac{1}{\cos\theta}+\frac{1+\cos\theta}{\sin\theta\cos\theta}\right)^2$

$=\frac{(1+\sin\theta+\cos\theta)^2}{\sin^2\theta\cos^2\theta}$

$=\frac{2+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}{\sin^2\theta\cos^2\theta};$

$\therefore$ First side $=2\sec^2\theta\frac{(1-\cos\theta)}{\sin^2\theta}(1+\sin\theta+\cos\theta+\sin\theta\cos\theta)$

$=\frac{2\sec^2\theta(1+\cos\theta)(1+\sin\theta)}{1+\cos\theta}$

$=2\sec^2\theta(1+\sin\theta).$

224. The relation given will be true if

$$\frac{1}{a+b+c}-\frac{1}{a+c}=\frac{1}{b+c}-\frac{2}{a+b+c};$$

i.e. if $$-\frac{b}{a+c}=\frac{a-b-c}{b+c},\text{ or }\frac{b}{a+c}=\frac{b+c-a}{b+c},$$

i.e. if $$b(b+c)=(b+c)(a+c)-a^2-ac.$$

From this we easily deduce $\dfrac{a^2+b^2-c^2}{ab}=1$, which is true when $C=60°$.

225. The solution of this example is merely an extension of that of Ex. 205.

226. We have $m\sin(\alpha-\theta)\cos(\alpha-\theta)=n\sin\theta\cos\theta$,

$$m\sin 2(\alpha-\theta)=n\sin 2\theta;$$

$$\therefore\ \frac{\sin 2(\alpha-\theta)-\sin 2\theta}{\sin 2(\alpha-\theta)+\sin 2\theta}=\frac{n-m}{n+m},$$

$$\frac{\sin(\alpha-2\theta)\cos\alpha}{\cos(\alpha-2\theta)\sin\alpha}=\frac{n-m}{n+m},$$

$$\tan(\alpha-2\theta)=\frac{n-m}{n+m}\tan\alpha,$$

$$\alpha-2\theta=\tan^{-1}\left(\frac{n-m}{n+m}\tan\alpha\right),$$

$$\theta=\frac{1}{2}\left\{\alpha-\tan^{-1}\left(\frac{n-m}{n+m}\tan\alpha\right)\right\}.$$

227. Put k for each of the equal ratios, then it easily follows that $s=k(1+n^2)$.

Now $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{(1-m^2)m^2(1+n^2)}{(1+n^2)n^2(1-m^2)}}=\frac{m}{n};$$

$$\therefore A=2\tan^{-1}\frac{m}{n};\text{ similarly } B=2\tan^{-1}mn.$$

Again, $$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=k^2\sqrt{(1+n^2)^2(1-m^2)^2m^2n^2}$$

$$=k^2(1-m^2)(1+n^2)mn=kcmn$$

$$=\frac{mnbc}{m^2+n^2},\text{ since }\frac{b}{m^2+n^2}=k.$$

228. See figure and solution of Example II. page 190.

Here $$h=CD=\frac{a\sin\beta}{\cos(2\alpha+\beta)},$$

$$l=DE=\frac{a\sin\alpha\cos(\alpha+\beta)}{\cos(2\alpha+\beta)}.$$

But $2\alpha+\beta+\theta=90°$; $\therefore\ \cos(2\alpha+\beta)=\sin\theta$;

$\therefore$ by substitution, $h=a\sin\beta\operatorname{cosec}\theta$,

$$\begin{aligned}2l&=2a\operatorname{cosec}\theta\sin\alpha\cos(\alpha+\beta)\\&=a\operatorname{cosec}\theta\{\sin(2\alpha+\beta)-\sin\beta\}\\&=a\operatorname{cosec}\theta(\cos\theta-\sin\beta).\end{aligned}$$

229. We have

$$\begin{aligned}&\frac{1}{2}\tan\frac{\theta}{2}+\cot\theta\\&=\frac{1}{2}\cdot\frac{1-\cos\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}=\frac{1-\cos\theta+2\cos\theta}{2\sin\theta}=\frac{1+\cos\theta}{2\sin\theta}\\&=\frac{2\cos^2\frac{\theta}{2}}{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\\&=\frac{\cos^2\frac{\theta}{4}-\sin^2\frac{\theta}{4}}{4\sin\frac{\theta}{4}\cos\frac{\theta}{4}}=\frac{1}{4}\cot\frac{\theta}{4}-\frac{1}{4}\tan\frac{\theta}{4}.\end{aligned}$$

230. With the figure of Art. 214 we have $AD=\frac{AO}{2m}$.

Let $\theta=\angle AOD$, then $\cos AOB=1-2\sin^2\theta$

$$=1-\frac{2AD^2}{AO^2}=1-\frac{1}{2m^2}=\frac{2m^2-1}{2m^2};$$

$$\therefore\ AOB=\sec^{-1}\frac{2m^2}{2m^2-1}.$$

231. With the figure of Art 268, Ex. 1, we have

$$\frac{PN}{ON}=\tan 1''=\text{radian measure of } 1'', \text{ approx.}$$

$$=\frac{\pi}{180}\times\frac{1}{60}\times\frac{1}{60};$$

$$\therefore\ ON=\frac{180\times60\times60}{\pi}\text{ inches}$$

$$=\frac{180\times60\times60}{1760\times3\times12\pi}\text{ miles}$$

$$=\frac{1800}{176\pi}\text{ miles}=3\tfrac{1}{4}\text{ miles, nearly.}$$

232. $$\tan^{-1}y=2\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{\dfrac{4x}{1-x^2}}{1-\dfrac{4x^2}{(1-x^2)^2}}$$

$$=\tan^{-1}\frac{4x(1-x^2)}{1-6x^2+x^4};$$

$$\therefore\ y=\frac{4x(1-x^2)}{1-6x^2+x^4}.$$

If $$y=\tan\frac{\pi}{2},\quad 1-6x^2+x^4=0,$$

but $$x=\tan\frac{1}{4}(\tan^{-1}y)=\tan\frac{\pi}{8},$$

thus $\tan\frac{\pi}{8}$ is a root of $x^4-6x^2+1=0$.

233. We have $$\tan^2\alpha=\frac{1-\cos2\alpha}{1+\cos2\alpha}=\frac{49}{529};$$

$$\therefore\ \tan\alpha=\pm\frac{7}{23}.$$

The two values may be explained as in Art. 261, Ex. 2.

234. We have $$\frac{\sin\theta}{a}=\frac{\sin\phi}{b}=\frac{\sin(\theta+\phi)}{c};$$

$$\therefore\ \frac{\sin\theta+\sin\phi}{a+b}=\frac{\sin(\theta+\phi)}{c}.$$

But $$a+b=2c,$$

$$\therefore\ \sin\theta+\sin\phi=2\sin(\theta+\phi),$$

whence $$\cos\frac{\theta-\phi}{2}=2\cos\frac{\theta+\phi}{2}\quad\text{.....(1),}$$

or $$\cos\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}-\cos\frac{\theta+\phi}{2}$$

$$=2\sin\frac{\theta}{2}\sin\frac{\phi}{2}\quad\text{.....(2).}$$

Now $$c=(a+b)\sin\frac{C}{2}\sec\phi=395\sin\frac{C}{2}\sec\phi,$$

$$\log 395=\log 79+1-\log 2$$
$$=2\cdot59660$$
$$\log\sin\frac{C}{2}=\bar{1}\cdot73998$$
$$\log\sec\phi=\ \cdot01920$$
$$\log c=\overline{2}\cdot35578$$
$$\therefore\ c=226\cdot87.$$

205. $$2\cos^2 2\theta=1+\cos 4\theta;$$
$$\therefore\ 2\cos 2\theta=\sqrt{2+2\cos 4\theta}.$$

Similarly $$2\cos\theta=\sqrt{2+2\cos 2\theta}=\sqrt{2+\sqrt{2+2\cos 4\theta}}.$$

206. Let $$\sin^{-1}\frac{3}{\sqrt{73}}=\alpha,\ \text{so that}\ \cos\alpha=\frac{8}{\sqrt{73}},$$

and let $$\cos^{-1}\frac{11}{\sqrt{146}}=\beta,\ \text{so that}\ \sin\beta=\frac{5}{\sqrt{146}}.$$

Then $$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$=\frac{3}{\sqrt{73}}\cdot\frac{11}{\sqrt{146}}+\frac{8}{\sqrt{73}}\cdot\frac{5}{\sqrt{146}}=\frac{73}{73\sqrt{2}}=\frac{1}{\sqrt{2}}$$
$$=\sin\frac{\pi}{4}=\sin\left(\frac{5\pi}{12}-\frac{\pi}{6}\right);$$
$$\therefore\ \alpha+\beta=\frac{5\pi}{12}-\frac{\pi}{6}=\frac{5\pi}{12}-\sin^{-1}\frac{1}{2}.$$

Again $$\tan^{-1}\frac{\frac{x-1}{x+1}+\frac{2x-1}{2x+1}}{1-\frac{x-1}{x+1}\cdot\frac{2x-1}{2x+1}}=\tan^{-1}\frac{23}{36};$$
$$\tan^{-1}\frac{4x^2-2}{6x}=\tan^{-1}\frac{23}{36};$$
$$\therefore\ 36(2x^2-1)=69x,$$

or $$24x^2-23x-12=0;$$
$$\therefore\ (3x-4)(8x+3)=0,$$

that is, $$x=\frac{4}{3},\ \text{or}\ -\frac{3}{8}.$$

207. We have $x=\frac{2\Delta}{a}$, $R=\frac{abc}{4\Delta}$;

$$\therefore\ x=\frac{1}{a}\cdot\frac{abc}{2R};\ \therefore\ \frac{bx}{c}=\frac{b^2}{2R},$$

that is, $$\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}=\frac{a^2+b^2+c^2}{2R}.$$

208. We have $\cos(\alpha+\theta)=\cos\left\{\frac{\pi}{2}-(\alpha-\theta)\right\}$,

$$\therefore\ \alpha+\theta=2m\pi\pm\left\{\frac{\pi}{2}-(\alpha-\theta)\right\};$$

the upper sign gives $$2\alpha=2m\pi+\frac{\pi}{2},$$

and the lower sign gives $$2\theta=2m\pi-\frac{\pi}{2}.$$

209. With the notation of Art. 228,

$$SI^2=R^2-2Rr.$$

If a be the base of the triangle, $A=120°$,

$$B=C=30°;\ \therefore\ r=4R\sin 60°\sin 15°\sin 15°;$$

$$\therefore\ SI^2=R^2-8R^2\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2$$

$$=R^2(4-2\sqrt{3});$$

$$\therefore\ SI=R(\sqrt{3}-1)$$

$$=\frac{a}{2\sin A}(\sqrt{3}-1)=\frac{a(\sqrt{3}-1)}{\sqrt{3}},$$

$$\therefore\ SI:a=\sqrt{3}-1:\sqrt{3}.$$

210. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$=1+\frac{r}{R}=1+\frac{3}{4},$$

$$\therefore\ 4(\cos A+\cos B+\cos C)=7.$$

211. Since $$\frac{\cos(\theta-\alpha)}{\sin(\theta+\alpha)}=\frac{1+m}{1-m},$$

dividendo and componendo, we have

$$\frac{\cos(\theta-\alpha)-\sin(\theta+\alpha)}{\cos(\theta-\alpha)+\sin(\theta+\alpha)}=m.$$

By expanding the sines and cosines we obtain

$$\frac{(\cos\theta-\sin\theta)(\cos\alpha-\sin\alpha)}{(\cos\theta+\sin\theta)(\cos\alpha+\sin\alpha)}=m,$$

or $$\frac{1-\tan\theta}{1+\tan\theta}=m\left(\frac{\cot\alpha+1}{\cot\alpha-1}\right).$$ [See XI. *b*. Ex. 6, 7.]

212. (1) $$\sin 5\theta-\sin 3\theta=\sqrt{2}\cos 4\theta;$$

$$2\cos 4\theta\sin\theta=\sqrt{2}\cos 4\theta;$$

$$\therefore\ \cos 4\theta=0;\ \text{whence } 4\theta=(2n+1)\frac{\pi}{2},$$

or $$\sin\theta=\frac{1}{\sqrt{2}};\ \text{whence } \theta=n\pi+(-1)^n\frac{\pi}{4}.$$

(2) $$1+\sin 2\theta=\frac{1+\tan\theta}{1-\tan\theta}.$$

$$1+\frac{2\tan\theta}{1+\tan^2\theta}=\frac{1+\tan\theta}{1-\tan\theta}.$$

$$(1+\tan\theta)^2(1-\tan\theta)=(1+\tan\theta)(1+\tan^2\theta);$$

$$\therefore\ 1+\tan\theta=0;\ \text{whence } \theta=n\pi+\frac{3\pi}{4},$$

or $$1-\tan^2\theta=1+\tan^2\theta;$$

$$\therefore\ \tan\theta=0;\ \text{whence } \theta=n\pi.$$

213. We have $$2\cos\frac{A+B}{2}\cos\frac{A-B}{2}=4\sin^2\frac{C}{2};$$

$$\therefore\ \cos\frac{A-B}{2}=2\sin\frac{C}{2},\ \text{or } 2\cos\frac{A-B}{2}\sin\frac{A+B}{2}=4\sin\frac{C}{2}\cos\frac{C}{2};$$

that is, $$\sin A+\sin B=2\sin C,\ \text{or } a+b=2c.$$

214. With the figure on p. 186, we have $\tan\beta=\frac{1}{9}$, $PA=80$ ft. $CA=100$ ft. Let $BP=x$ ft., then

$$\tan\theta=\frac{x+80}{100},\ \tan(\theta-\beta)=\frac{80}{100}=\frac{4}{5},$$

$$\therefore\ \frac{\tan\theta-\tan\beta}{1+\tan\theta\tan\beta}=\frac{4}{5},$$

$$\frac{\frac{x+80}{100}-\frac{1}{9}}{1+\frac{x+80}{900}}=\frac{4}{5};$$

$$\therefore\ 5(9x+720-100)=4(980+x);$$

$$45x+3100=3920+4x;$$

$$41x=820;\ \text{or } x=20.$$

215. (1) $\cot^{-1}7+\cot^{-1}8=\cot^{-1}\dfrac{7.8-1}{7+8}=\cot^{-1}\dfrac{55}{15}$,

$$\cot^{-1}3-\cot^{-1}18=\cot^{-1}\frac{3.18+1}{18-3}=\cot^{-1}\frac{55}{15}.$$

(2) $$4\tan^{-1}\frac{1}{5}=2\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{5^2}}=2\tan^{-1}\frac{5}{12}$$

$$=\tan^{-1}\frac{\frac{2\times5}{12}}{1-\frac{5^2}{12^2}}=\tan^{-1}\frac{120}{119}.$$

$$\therefore\ 4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\tan^{-1}\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}=\tan^{-1}\frac{120.239-119}{119.239+120}$$

$$=\tan^{-1}\frac{119.239+(239-119)}{119.239+120}=\tan^{-1}1=\frac{\pi}{4}.$$

216. Proceeding as in Art. 259, Ex. 2, we find that $\frac{A}{2}$ lies between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$; that is A lies between $(8n+3)\frac{\pi}{2}$ and $(8n+5)\frac{\pi}{2}$.

217. First side $=\frac{1}{2}\left\{1-\cos\left(\frac{\pi}{4}+\theta\right)-1+\cos\left(\frac{\pi}{4}-\theta\right)\right\}$

$$=\sin\frac{\pi}{4}\sin\theta=\frac{1}{\sqrt{2}}\sin\theta.$$

218. From the two given relations we easily deduce

$$x=\frac{\sin\theta}{\sin(\theta+\phi)},\quad y=\frac{\sin\phi}{\sin(\theta+\phi)},$$

$$\therefore\ \sin\theta:\sin\phi=x:y.$$

219.
$$\tan^{-1}\frac{x+1+x-1}{1-(x^2-1)}=\tan^{-1}\frac{8}{31};$$

$$\therefore\ \frac{2x}{2-x^2}=\frac{8}{31},\ \text{or}\ 4x^2+31x-8=0;$$

$$\therefore\ (4x-1)(x+8)=0,\ \text{or}\ x=\frac{1}{4},\ \text{or}\ -8.$$

Again $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)$

$=1+\tan^2(\tan^{-1}2)+1+\cot^2(\cot^{-1}3)$

$=1+4+1+9=15.$

220. Let $5A=\alpha$, $5B=\beta$, $5C=\gamma$, then $\alpha+\beta+\gamma=5\pi$;

$\therefore \sin(\alpha+\beta)=\sin(\pi-\gamma)=\sin\gamma$; $\cos\gamma=-\cos(\alpha+\beta)$,

$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=2\sin(\alpha+\beta)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma$

$=2\sin\gamma\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}$

$=4\sin\alpha\sin\beta\sin\gamma.$

In the second case the sum of the three angles is $\frac{16\pi}{2^5}$, or $\frac{\pi}{2}$ and the result easily follows as in Art. 135, Ex. 2.

221. See solution to XVIII. *a*. Ex. 24.

222. We have $x=\frac{BD\sin 15^\circ}{\sin 50^\circ}$; $BD=\frac{100}{\cos 25^\circ}$;

$\therefore x=\frac{100\cos 75^\circ}{\cos 40^\circ\cos 25^\circ}$;

$\therefore \log x=2+\log\cos 75^\circ-(\log\cos 40^\circ+\log\cos 25^\circ)$

$=1\cdot4129962-(\bar{1}\cdot8415297)$

$=1\cdot5714665$

$\log 37\cdot279=1\cdot5714643$ prop^{l}. increase $=\frac{22}{116}\times\cdot001$

diff. 22 $=\cdot00019;$

$\therefore x=37\cdot27919.$

223. $\{\sec\theta+\operatorname{cosec}\theta(1+\sec\theta)\}^2=\left(\frac{1}{\cos\theta}+\frac{1+\cos\theta}{\sin\theta\cos\theta}\right)^2$

$=\frac{(1+\sin\theta+\cos\theta)^2}{\sin^2\theta\cos^2\theta}$

$=\frac{2+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta}{\sin^2\theta\cos^2\theta};$

$\therefore$ First side $=2\sec^2\theta\frac{(1-\cos\theta)}{\sin^2\theta}(1+\sin\theta+\cos\theta+\sin\theta\cos\theta)$

$=\frac{2\sec^2\theta(1+\cos\theta)(1+\sin\theta)}{1+\cos\theta}$

$=2\sec^2\theta(1+\sin\theta).$

224. The relation given will be true if

$$\frac{1}{a+b+c}-\frac{1}{a+c}=\frac{1}{b+c}-\frac{2}{a+b+c};$$

i.e. if $$-\frac{b}{a+c}=\frac{a-b-c}{b+c}, \text{ or } \frac{b}{a+c}=\frac{b+c-a}{b+c},$$

i.e. if $$b(b+c)=(b+c)(a+c)-a^2-ac.$$

From this we easily deduce $\dfrac{a^2+b^2-c^2}{ab}=1$, which is true when $C=60°$.

225. The solution of this example is merely an extension of that of Ex. 205.

226. We have $m\sin(\alpha-\theta)\cos(\alpha-\theta)=n\sin\theta\cos\theta$,

$$m\sin 2(\alpha-\theta)=n\sin 2\theta;$$

$$\therefore \frac{\sin 2(\alpha-\theta)-\sin 2\theta}{\sin 2(\alpha-\theta)+\sin 2\theta}=\frac{n-m}{n+m},$$

$$\frac{\sin(\alpha-2\theta)\cos\alpha}{\cos(\alpha-2\theta)\sin\alpha}=\frac{n-m}{n+m},$$

$$\tan(\alpha-2\theta)=\frac{n-m}{n+m}\tan\alpha,$$

$$\alpha-2\theta=\tan^{-1}\left(\frac{n-m}{n+m}\tan\alpha\right),$$

$$\theta=\frac{1}{2}\left\{\alpha-\tan^{-1}\left(\frac{n-m}{n+m}\tan\alpha\right)\right\}.$$

227. Put k for each of the equal ratios, then it easily follows that $s=k(1+n^2)$.

Now $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{(1-m^2)m^2(1+n^2)}{(1+n^2)n^2(1-m^2)}}=\frac{m}{n};$$

$$\therefore A=2\tan^{-1}\frac{m}{n}; \text{ similarly } B=2\tan^{-1}mn.$$

Again, $$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$

$$=k^2\sqrt{(1+n^2)^2(1-m^2)^2m^2n^2}$$

$$=k^2(1-m^2)(1+n^2)mn=kcmn$$

$$=\frac{mnbc}{m^2+n^2}, \text{ since } \frac{b}{m^2+n^2}=k.$$

228. See figure and solution of Example II. page 190.

Here $$h=CD=\frac{a\sin\beta}{\cos(2\alpha+\beta)},$$

$$l=DE=\frac{a\sin\alpha\cos(\alpha+\beta)}{\cos(2\alpha+\beta)}.$$

But $2\alpha+\beta+\theta=90°$; $\therefore\ \cos(2\alpha+\beta)=\sin\theta$;

$\therefore$ by substitution, $h=a\sin\beta\operatorname{cosec}\theta$,

$$\begin{aligned}2l&=2a\operatorname{cosec}\theta\sin\alpha\cos(\alpha+\beta)\\&=a\operatorname{cosec}\theta\{\sin(2\alpha+\beta)-\sin\beta\}\\&=a\operatorname{cosec}\theta(\cos\theta-\sin\beta).\end{aligned}$$

229. We have

$$\begin{aligned}&\frac{1}{2}\tan\frac{\theta}{2}+\cot\theta\\&=\frac{1}{2}\cdot\frac{1-\cos\theta}{\sin\theta}+\frac{\cos\theta}{\sin\theta}=\frac{1-\cos\theta+2\cos\theta}{2\sin\theta}=\frac{1+\cos\theta}{2\sin\theta}\\&=\frac{2\cos^2\frac{\theta}{2}}{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}\\&=\frac{\cos^2\frac{\theta}{4}-\sin^2\frac{\theta}{4}}{4\sin\frac{\theta}{4}\cos\frac{\theta}{4}}=\frac{1}{4}\cot\frac{\theta}{4}-\frac{1}{4}\tan\frac{\theta}{4}.\end{aligned}$$

230. With the figure of Art. 214 we have $AD=\frac{AO}{2m}$.

Let $\theta=\angle AOD$, then $\cos AOB=1-2\sin^2\theta$

$$=1-\frac{2AD^2}{AO^2}=1-\frac{1}{2m^2}=\frac{2m^2-1}{2m^2};$$

$$\therefore\ AOB=\sec^{-1}\frac{2m^2}{2m^2-1}.$$

231. With the figure of Art 268, Ex. 1, we have

$$\frac{PN}{ON}=\tan 1''=\text{radian measure of } 1'', \text{ approx.}$$

$$=\frac{\pi}{180}\times\frac{1}{60}\times\frac{1}{60};$$

$$\therefore\ ON=\frac{180\times60\times60}{\pi}\text{ inches}$$

$$=\frac{180\times60\times60}{1760\times3\times12\pi}\text{ miles}$$

$$=\frac{1800}{176\pi}\text{ miles}=3\tfrac{1}{4}\text{ miles, nearly.}$$

232. $$\tan^{-1}y=2\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{\dfrac{4x}{1-x^2}}{1-\dfrac{4x^2}{(1-x^2)^2}}$$

$$=\tan^{-1}\frac{4x(1-x^2)}{1-6x^2+x^4};$$

$$\therefore\ y=\frac{4x(1-x^2)}{1-6x^2+x^4}.$$

If $$y=\tan\frac{\pi}{2},\quad 1-6x^2+x^4=0,$$

but $$x=\tan\frac{1}{4}(\tan^{-1}y)=\tan\frac{\pi}{8},$$

thus $\tan\frac{\pi}{8}$ is a root of $x^4-6x^2+1=0$.

233. We have $$\tan^2\alpha=\frac{1-\cos2\alpha}{1+\cos2\alpha}=\frac{49}{529};$$

$$\therefore\ \tan\alpha=\pm\frac{7}{23}.$$

The two values may be explained as in Art. 261, Ex. 2.

234. We have $$\frac{\sin\theta}{a}=\frac{\sin\phi}{b}=\frac{\sin(\theta+\phi)}{c};$$

$$\therefore\ \frac{\sin\theta+\sin\phi}{a+b}=\frac{\sin(\theta+\phi)}{c}.$$

But $$a+b=2c,$$

$$\therefore\ \sin\theta+\sin\phi=2\sin(\theta+\phi),$$

whence $$\cos\frac{\theta-\phi}{2}=2\cos\frac{\theta+\phi}{2}\quad\text{.....(1),}$$

or $$\cos\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}-\cos\frac{\theta+\phi}{2}$$

$$=2\sin\frac{\theta}{2}\sin\frac{\phi}{2}\quad\text{.....(2).}$$

Now $\cos\theta+\cos\phi=2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$

$=2\left(2\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right)2\cos\frac{\theta+\phi}{2}$, by (1) and (2)

$=4\sin\frac{\theta}{2}\sin\frac{\phi}{2}.4\sin\frac{\theta}{2}\sin\frac{\phi}{2}$, by (2)

$=16\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}$

$=4(1-\cos\theta)(1-\cos\phi)$.

235. (1) $\sin 7\theta+\sin\theta=\sin 4\theta$;

$\therefore\ 2\sin 4\theta\cos 3\theta=\sin 4\theta$;

$\therefore$ either $\sin 4\theta=0$, or $\cos 3\theta=\frac{1}{2}$;

that is, $4\theta=n\pi$, or $3\theta=2n\pi\pm\frac{\pi}{3}$.

(2) $\tan x-\frac{\sqrt{3}}{\tan x}+1-\sqrt{3}=0$;

$\tan^2 x-(\sqrt{3}-1)\tan x-\sqrt{3}=0$;

$(\tan x-\sqrt{3})(\tan x+1)=0$;

$\therefore$ either $\tan x=\sqrt{3}$, or $\tan x=-1$;

that is, $x=n\pi+\frac{\pi}{3}$, or $x=n\pi+\frac{3\pi}{4}$.

236. (1) $\sin 3A=\sin 3(180^\circ-\overline{B+C})$

$=\sin(360^\circ+180^\circ-3\overline{B+C})=\sin 3\overline{B+C}$.

We have only now to prove that

$$\Sigma\sin 3(B+C)\sin(B-C)=0,$$

and this follows by separating each term into the difference of two cosines.

(2) It will be sufficient to prove that

$$\Sigma\sin^3 A\sin(B-C)=0.$$

Now $\sin^3 A=\frac{3}{4}\sin A-\frac{1}{4}\sin 3A$;

$\therefore\ \Sigma\sin^3 A\sin(B-C)=\frac{3}{4}\Sigma\sin A\sin\overline{B-C}-\frac{1}{4}\Sigma\sin 3A\sin\overline{B-C}$

$=0$,

by the first part of the question.

237. The angle $ACP = \theta - A$.

$$\therefore \frac{AP}{PC} = \frac{\sin(\theta - A)}{\sin A} = \sin\theta \cot A - \cos\theta,$$

$$\frac{PC}{PB} = \frac{\sin B}{\sin(\theta + B)} = \frac{1}{\sin\theta \cot B + \cos\theta};$$

$\therefore$ by multiplication

$$\frac{AP}{PB} = \frac{m}{n} = \frac{\sin\theta \cot A - \cos\theta}{\sin\theta \cot B + \cos\theta};$$

whence $$\sin\theta (n \cot A - m \cot B) = (m+n)\cos\theta,$$

or $$(m+n)\cot\theta = n\cot A - m\cot B.$$

238. The equation may be written

$$a\sin\theta - \cos\theta + b = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since α and β are roots of this equation

$$a\sin\alpha - \cos\alpha + b = 0,$$

$$a\sin\beta - \cos\beta + b = 0,$$

whence a and b may be found.

Again from (1),

$$(a\sin\theta + b)^2 = 1 - \sin^2\theta,$$

or $$(1+a^2)\sin^2\theta + 2ab\sin\theta + b^2 - 1 = 0;$$

since α, β are roots of this equation,

$$\sin\alpha + \sin\beta = -\frac{2ab}{1+a^2}.$$

Similarly we may shew that $\cos\alpha + \cos\beta = \frac{2b}{1+a^2}$, whence the required result follows.

239. Write s and c for $\sin\theta$ and $\cos\theta$ respectively; then

$$\frac{u_3 - u_5}{u_1} = \frac{s^3 + c^3 - (s^5 + c^5)}{s + c} = \frac{s^3(1-s^2) + c^3(1-c^2)}{s+c} = \frac{s^3c^2 + c^3s^2}{s+c} = s^2c^2.$$

Again $$\frac{u_5 - u_7}{u_3} = \frac{s^5 + c^5 - (s^7 + c^7)}{s^3 + c^3} = \frac{s^5(1-s^2) + c^5(1-c^2)}{s^3 + c^3}$$

$$= \frac{s^5c^2 + c^5s^2}{s^3 + c^3} = s^2c^2.$$

240. Let E, F be the first and second points of observation respectively; then $EF=a$, and EAD is a straight line. Let $x=$a side of the square base, then $EA=AB=AD=x$. Then if $\angle AFD=\theta$, we have

$$AD^2=AF^2+FD^2-2AF\,.\,FD\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But $AD^2=x^2$, $AF^2=x^2+a^2$, $FD^2=4x^2+a^2$. Also $\cos\theta=\frac{1}{3}\sqrt{8}$. Substituting these values in (1) we obtain $x=\frac{a\sqrt{2}}{2}$.

241. (1) First side $=\dfrac{1-\sin^2 A}{\sin A}\,.\,\dfrac{1-\cos^2 A}{\cos A}=\dfrac{\cos^2 A\sin^2 A}{\sin A\cos A}$

$=\sin A\cos A.$

Again, second side $=\left(\dfrac{\sin^2 A+\cos^2 A}{\sin A\cos A}\right)^{-1}=\sin A\cos A.$

(2) First side $=\dfrac{\tan\theta}{\sec^4\theta}+\dfrac{\cot\theta}{\operatorname{cosec}^4\theta}$

$=\sin\theta\cos^3\theta+\cos\theta\sin^3\theta$

$=\sin\theta\cos\theta\,(\sin^2\theta+\cos^2\theta)$

$=\frac{1}{2}\sin 2\theta.$

242. We have $2\sin 4\theta\cos\theta=\frac{1}{2}+2\sin\frac{5\theta}{2}\cos\frac{5\theta}{2}$;

$\therefore\ \sin 5\theta+\sin 3\theta=\frac{1}{2}+2\sin\frac{5\theta}{2}\cos\frac{5\theta}{2}$;

$\therefore\ \sin 3\theta=\frac{1}{2}$,

one solution of which is $3\theta=30°$.

243. $\tan^{-1}\dfrac{2mn}{m^2-n^2}=\tan^{-1}\dfrac{2\frac{n}{m}}{1-\frac{n^2}{m^2}}=2\tan^{-1}\dfrac{n}{m}$;

$\therefore$ first side $=2\tan^{-1}\dfrac{n}{m}+2\tan^{-1}\dfrac{q}{p}$

$=2\left(\tan^{-1}\dfrac{n}{m}+\tan^{-1}\dfrac{q}{p}\right)$

$=2\tan^{-1}\dfrac{\frac{n}{m}+\frac{q}{p}}{1-\frac{nq}{mp}}$;

$\therefore$ first side $= 2\tan^{-1}\dfrac{np+mq}{mp-nq} = 2\tan^{-1}\dfrac{N}{M}$

$= \tan^{-1}\dfrac{2MN}{M^2-N^2}$.

244. Write $\dfrac{r}{s-a}$ for $\tan\dfrac{A}{2}$, then the first side becomes

$$\frac{r}{(s-a)(a-b)(a-c)} + \text{two similar terms.}$$

Now $$\frac{r}{(s-a)(a-b)(a-c)} = -\frac{rs(b-c)(s-b)(s-c)}{\Delta^2(a-b)(b-c)(c-a)}$$

$$= -\frac{1}{\Delta}\frac{(b-c)\{s^2-s(b+c)+bc\}}{(a-b)(b-c)(c-a)}.$$

Now $\Sigma(b-c)\{s^2-s(b+c)+bc\} = -(a-b)(b-c)(c-a)$.
[See Hall and Knight's Elem. Algebra, Art. 224.]

Thus the first side reduces to $\dfrac{1}{\Delta}$.

245. We have $$\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right) = 2;$$

or $$(b-a)(c-a) = 2(s-a)^2;$$

$$bc - ac - ab + a^2 = 2s^2 - 4as + 2a^2;$$

$$\therefore\ bc = 2s^2 - 4as + a(a+b+c)$$

$$= 2s^2 - 4as + 2as;$$

whence $$\frac{s(s-a)}{bc} = \frac{1}{2};$$

that is, $$\cos\frac{A}{2} = \frac{1}{\sqrt{2}}, \text{ or } A = 90°.$$

246. Let $A = 58°\,40'\,3{\cdot}9''$, $b = 237$, $c = 158$.

Then as in Art. 197 we obtain

$$a = (b+c)\sin\theta, \text{ where } \cos\theta = \frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2},$$

$$\cos\theta = \frac{2\sqrt{237\times 158}}{395}\cos 29°\,20'\,1{\cdot}95''$$

$$= \frac{2\sqrt{6}}{5}\cos 29°\,20'\,1{\cdot}95'',$$

$$\begin{array}{rl}
\log 2 = & \cdot 3010300 \\
\log 3 = & \cdot 4771213 \\
2 \mid & \cdot 7781513 \\
\frac{1}{2}\log 6 = & \cdot 3890757 \\
2\log 2 - 1 = & \bar{1}\cdot 6020600 \\
\log\cos 29^\circ 20' = & \bar{1}\cdot 9404091 \\
\textit{Subtract}\ \text{diff. for } 1\cdot 95'' & 23 \\
\log\cos\theta = & \bar{1}\cdot 9315425 \\
\log\cos 31^\circ 20' = & \bar{1}\cdot 9315374 \\
& 51
\end{array}$$

$\frac{51}{769} \times 60'' = 3\cdot 97''.$

$\therefore\ \theta = 31^\circ 19' 56''.$

Again

$$\begin{array}{rl}
\log 395 = & 2\cdot 5965971 \\
\log\sin 31^\circ 19' = & \bar{1}\cdot 7158092 \\
\text{diff. for } 56'' & 1937 \\
\log a = & 2\cdot 3126000 \\
\log 205\cdot 4 = & 2\cdot 3126004
\end{array}$$

$\therefore\ a = 205\cdot 4.$

247. We have $$\frac{\sin(A+B)}{\cos(A+B)} = \frac{3\sin A}{\cos A};$$

whence $$\sin(A+B)\cos A = 3\cos(A+B)\sin A,$$

or $$\sin(2A+B) + \sin B = 3\sin(2A+B) - 3\sin B;$$

that is $$2\sin(2A+B) = 4\sin B.$$

Multiply by $\cos B$; then by separating the product on the left into the sum of two sines we obtain the required result.

248. $$\begin{aligned}
\text{First side} &= 2\sin(\theta-\alpha)\{\sin(\theta-\alpha) + \sin(2m\theta - \alpha - \theta)\} \\
&= 2\sin^2(\theta-\alpha) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2\alpha) \\
&= 1 - \cos(2\theta - 2\alpha) + \cos(2\theta - 2m\theta) - \cos(2m\theta - 2\alpha).
\end{aligned}$$

249. Let AD be perpendicular to BC and meet the circum-circle in E; then $\angle BED = C$, and $a = DE$.

Now $$\frac{BD}{a} = \tan C, \text{ and } \frac{DC}{a} = \tan B;$$

$$\therefore\ \frac{BD+DC}{a} = \frac{a}{a} = \tan B + \tan C.$$

Similarly $$\frac{b}{\beta} = \tan C + \tan A, \quad \frac{c}{\gamma} = \tan A + \tan B;$$

whence the result follows.

250. From the first equation, $3\sin^2 A = 1 - 2\sin^2 B = \cos 2B$; and from the second equation,

$$6\sin A\cos A - 2\sin 2B = 0;$$

multiply each term by $\sin A$, and we have

$$3\sin^2 A\cos A - \sin 2B\sin A = 0.$$

Substituting $\cos 2B$ for $3\sin^2 A$, we obtain

$$\cos 2B\cos A - \sin 2B\sin A = 0;$$

that is, $\cos(A+2B) = 0$; or $A + 2B = 90°$.

251. (1) $\text{First side} = \cot^{-1}\left(\frac{1}{\cot 2x}\right) + \cot^{-1}\left(-\frac{1}{\cot 3x}\right)$

$$= \cot^{-1}\frac{\frac{1}{\cot 2x}\left(-\frac{1}{\cot 3x}\right) - 1}{-\frac{1}{\cot 3x} + \frac{1}{\cot 2x}}$$

$$= \cot^{-1}\left(\frac{\cot 3x\cot 2x + 1}{\cot 2x - \cot 3x}\right)$$

$$= \cot^{-1}(\cot x) = x.$$

(2) $\text{First side} = \tan^{-1}\frac{\frac{1-x}{1+x} - \frac{1-y}{1+y}}{1 + \frac{(1-x)(1-y)}{(1+x)(1+y)}} = \tan^{-1}\frac{2(y-x)}{2(1+xy)}$

$$= \sin^{-1}\frac{\frac{y-x}{1+xy}}{\sqrt{1 + \frac{(y-x)^2}{(1+xy)^2}}} = \sin^{-1}\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}.$$

252. See Art. 197.

Let A be the position of the station, B and C the positions of the two points; then $A = 49°\,45'$, $c = 1250$ yds., $b = 1575$ yds.

Now $a = 2825\cos\theta$, where $\sin\theta = \frac{2\sqrt{1250\times 1575}}{2825}\cos 24°\,52'\,30''$,

$\log 1250$	$= 3{\cdot}0969100$	$\log\cos 24°\,52' = \bar{1}{\cdot}9577456$	
$\log 1575$	$= 3{\cdot}1972806$	*subtract* $\frac{1}{2}\times 586$	293
	$2 \mid 6{\cdot}2941906$		$\bar{1}{\cdot}9577163$
	$3{\cdot}1470953$		
$\log 2$	$= \cdot 3010300$		
$\log\cos 24°\,52'\,30''$	$= \bar{1}{\cdot}9577163$		
	$3{\cdot}4058416$		
$\log 2825$	$= 3{\cdot}4510185$		
$\log\sin\theta$	$= \bar{1}{\cdot}9548231 = \log\sin 64°\,19'$.		

Again

$$\begin{array}{rr} \log 2825 & = 3{\cdot}4510185 \\ \log \cos 64° 19' & = \bar{1}{\cdot}6368859 \\ \log a & = 3{\cdot}0879044 \\ \log 1224{\cdot}3 & = 3{\cdot}0878878 \\ & \overline{166} \\ 4 & \overline{142} \\ & \overline{240} \\ 7 & \overline{249} \end{array}$$

$\therefore a = 1224{\cdot}347$ yards.

254. Multiply all through by 2; then

First side $= 1 + \cos 2S + 1 + \cos 2(S - A) +$ two similar terms

$= 4 + 2\cos(2S - A)\cos A + 2\cos(2S - B - C)\cos(B - C)$

$= 4 + 2\cos(B + C)\cos A + 2\cos A \cos(B - C)$

$= 4 + 4\cos A \cos B \cos C.$

255. It is easy to see that this is the same as Example 1 in Art. 135.

256. We have $R = \dfrac{a}{2\sin A} = 18 \operatorname{cosec} 61° 15'.$

$$\begin{array}{rr} \log 18 & = 1{\cdot}2552725 \\ \log \operatorname{cosec} 61° 15' & = \cdot 0571357 \\ \log R & = \bar{1}{\cdot}3124082 \\ \log 20{\cdot}530 & = 1{\cdot}3123889 \\ & \overline{193} \\ 9 & \overline{191} \end{array}$$

$\therefore R = 20{\cdot}5309.$

Again, $r = 4R \sin\dfrac{A}{2} \sin\dfrac{B}{2} \sin\dfrac{C}{2}.$

$$\begin{array}{rr} \log R & = 1{\cdot}3124082 \\ \log 4 & = \cdot 6020600 \\ \log \sin 30° 37' & = \bar{1}{\cdot}7069667 \\ \text{diff. for } 30'' & = \quad 1067 \\ \log \sin 36° 37' & = \bar{1}{\cdot}7755801 \\ \text{diff. for } 30'' & = \quad 850 \\ \log \sin 22° 45' & = \bar{1}{\cdot}5873865 \\ \therefore \log r & = \cdot 9845932 \\ \log 9{\cdot}6514 & = \cdot 9845903 \\ & \overline{29} \\ 6 & \overline{27} \end{array}$$

$\therefore r = 9{\cdot}65146.$

257. This follows from XVIII. *c*. Ex. 5 and XII. *d*. Ex. 12.

258. Let $\angle APB=\alpha,\ \angle BPC=\beta,\ \angle PBC=\gamma;$

then $$\frac{PB}{AB}=\frac{\sin(\gamma-\alpha)}{\sin\alpha},\qquad \frac{PB}{BC}=\frac{\sin(\beta+\gamma)}{\sin\beta};$$

but $$AB=BC,$$

$$\therefore\ \frac{\sin(\gamma-\alpha)}{\sin\alpha}=\frac{\sin(\beta+\gamma)}{\sin\beta},$$

or $$\sin\gamma\cot\alpha-\cos\gamma=\cos\gamma+\sin\gamma\cot\beta;$$

$$\therefore\ 2\cos\gamma=\sin\gamma(\cot\alpha-\cot\beta),$$

$$2\cot\gamma=\cot\alpha-\cot\beta;$$

that is, $$\frac{2}{T}=\frac{1}{t'}-\frac{1}{t},$$

since γ is the supplement of the angle BP makes with the road.

259. First side $$=\frac{(\cos B+\cos C)(1+2\cos A)}{(1+2\cos A)(1-\cos A)}=\frac{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin^2\frac{A}{2}}$$

$$=\frac{2\cos\frac{B-C}{2}}{2\sin\frac{A}{2}}\cdot\frac{\cos\frac{A}{2}}{\cos\frac{A}{2}}=\frac{2\cos\frac{B-C}{2}\sin\frac{B+C}{2}}{\sin A}$$

$$=\frac{\sin B+\sin C}{\sin A}=\frac{b+c}{a}.$$

260. From the fig. of Art. 219, we have $\frac{AI}{AI_1}=\frac{s-a}{s}$;

$$\therefore\ \text{first side}=\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}=\frac{3s-2s}{s}=1$$

261. Let $$\sin^{-1}\frac{1}{3}=\alpha,\ \text{then}\ \cos\alpha=\frac{2\sqrt{2}}{3};$$

$$\sin^{-1}\frac{3}{\sqrt{11}}=\beta,\ \text{then}\ \cos\beta=\frac{\sqrt{2}}{\sqrt{11}}.$$

Now $$\sin(\alpha+\beta)=\frac{1}{3}\cdot\frac{\sqrt{2}}{\sqrt{11}}+\frac{2\sqrt{2}}{3}\cdot\frac{3}{\sqrt{11}}=\frac{7\sqrt{2}}{3\sqrt{11}}$$

$$=\cos\left(\sin^{-1}\frac{1}{3\sqrt{11}}\right),$$

$$\therefore\ \alpha+\beta=\frac{\pi}{2}-\sin^{-1}\frac{1}{3\sqrt{11}},$$

or $$\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{1}{3\sqrt{11}}+\sin^{-1}\frac{3}{\sqrt{11}}=\frac{\pi}{2}.$$

262. We have $\cot\alpha(\cot\beta\cot\gamma-1)=\cot\beta+\cot\gamma$,

$$\therefore \cot(\beta+\gamma)=\frac{1}{\cot\alpha}=\frac{1}{\tan\alpha}$$

$$=\cot\left(\frac{\pi}{2}-\alpha\right);$$

$$\therefore \beta+\gamma=n\pi+\frac{\pi}{2}-\alpha,$$

or
$$\alpha+\beta+\gamma=(2n+1)\frac{\pi}{2}.$$

263. Since $\tan A+\tan B+\tan C=\tan A\tan B\tan C$,

the first side $=\dfrac{\tan^2 A+\tan^2 B+\tan^2 C}{\tan A+\tan B+\tan C}$

$$=\frac{(\tan A+\tan B+\tan C)^2-2\tan A\tan B-2\tan B\tan C-2\tan C\tan A}{\tan A+\tan B+\tan C}$$

$$=\tan A+\tan B+\tan C-\frac{2(\tan A\tan B+\ldots+\ldots)}{\tan A\tan B\tan C}$$

$$=\tan A+\tan B+\tan C-2(\cot A+\cot B+\cot C).$$

264. Let O_1, O_2 be the two points of observation, A and B the two objects, so that $\angle AO_1O_2=45°$, $AO_2O_1=O_1O_2B=22\frac{1}{2}°$. Then $\angle O_1AO_2=112\frac{1}{2}°$, $\angle O_1BO_2=22\frac{1}{2}°$, and $O_1B=O_1O_2=1$ mile.

Now from the $\triangle O_1AO_2$, $\dfrac{O_1A}{1\text{ mile}}=\dfrac{\sin 22\frac{1}{2}°}{\sin 112\frac{1}{2}°}=\tan 22\frac{1}{2}°=\sqrt{2}-1$;

$$\therefore O_1A=\sqrt{2}-1\text{ miles};\ \therefore AB=\sqrt{2}\text{ miles}.$$

Again, if p_1, p_2 be the perpendiculars from AB on O_1O_2

$$p_1+p_2=(O_1A+O_1B)\sin 45°=AB\sin 45°=1\text{ mile}.$$

265. This follows from the identity

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C,$$

where $A+B+C=180°$, by putting $A=20°$, $B=40°$, $C=120°$.

266. The equation may be written

$$(2\operatorname{cosec}2\theta)^3=3(2\operatorname{cosec}2\theta)+\frac{\cos^3\theta}{\sin^3\theta},$$

or
$$\left(\frac{1}{\sin\theta\cos\theta}\right)^3=\frac{3}{\sin\theta\cos\theta}+\frac{\cos^3\theta}{\sin^3\theta};$$

that is,
$$1=3\sin^2\theta\cos^2\theta+\cos^6\theta,$$

which reduces to $(\cos^2\theta-1)^3=0$, whence $\theta=n\pi$.

267. When $A+B+C=180°$,

$$1-\cos^2 B+\cos^2 A-\cos^2 C=\sin^2 B+\sin^2 C-\sin^2 A$$
$$=\sin^2 B+\sin(C+A)\sin(C-A)$$
$$=\sin B\{\sin(C+A)+\sin(C-A)\},$$
since $C+A=180°-B$,
$$=2\sin B\sin C\cos A.$$

When $A+B+C=0$,

$$1-\cos^2 B+\cos^2 A-\cos^2 C=\sin^2 B+\sin(C+A)\sin(C-A)$$
$$=-\sin B\{\sin(C+A)+\sin(C-A)\},$$
since $C+A=-B$,
$$=-2\sin B\sin C\cos A.$$

268. We have $\cot A-\cot B=\cot B-\cot C$,

that is
$$\frac{\sin(A-B)}{\sin A\sin B}=\frac{\sin(B-C)}{\sin B\sin C},$$

or
$$\frac{\sin(A-B)}{\sin(B+C)}=\frac{\sin(B-C)}{\sin(A+B)}.$$

Whence $\sin(A+B)\sin(A-B)=\sin(B+C)\sin(B-C)$,

$$\sin^2 A-\sin^2 B=\sin^2 B-\sin^2 C;$$

that is,
$$a^2-b^2=b^2-c^2.$$

269. Second side $=2\cos\frac{5\alpha-2\beta-\gamma}{4}\left\{\cos\frac{4\beta+3\gamma-3\alpha}{4}+\cos\frac{6\beta-7\gamma+\alpha}{4}\right\}$

$$=\cos\frac{2\alpha+2\beta+2\gamma}{4}+\cos\frac{6\beta+4\gamma-8\alpha}{4}+\cos\frac{6\alpha+4\beta-8\gamma}{4}$$
$$+\cos\frac{6\gamma+4\alpha-8\beta}{4}$$
$$=\cos\frac{\pi}{2}+\cos\left(\frac{3\beta}{2}+\gamma-2\alpha\right)+\cos\left(\frac{3\alpha}{2}+\beta-2\gamma\right)$$
$$+\cos\left(\frac{3\gamma}{2}+\alpha-2\beta\right)$$
$$=\text{first side}.$$

270. Denote the radii of the three escribed circles by x, y, z respectively, then we have to shew that

$$(y-z)(z-x)(x-y)+(y-z)(z+x)(x+y)+(z-x)(x+y)(y+z)$$
$$+(x-y)(y+z)(z+x)=0.$$

Taking the terms in pairs, the expression on the left reduces to

$$(y-z)\{2(zx+xy)\}+(y+z)\{2(zx-xy)\},$$

or
$$2x(y-z)(y+z)+2x(y+z)(z-y),$$

which is identically equal to zero.

271. We have $32 \sin\frac{A}{2}\sin\frac{5A}{2} = 16(\cos 2A - \cos 3A)$.

Now $\cos 2A = 2\cos^2 A - 1 = \frac{2\times 9}{16} - 1 = \frac{1}{8}$,

$\cos 3A = 4\cos^3 A - 3\cos A = \frac{4\times 27}{64} - \frac{3\times 3}{4} = -\frac{9}{16}$;

$\therefore\ 32 \sin\frac{A}{2}\sin\frac{5A}{2} = 16\left(\frac{1}{8} + \frac{9}{16}\right) = 11.$

272. Solving the quadratic, we have $\tan\theta = -1 \pm \sqrt{2}$.

Now $\sqrt{2} - 1 = \tan\frac{\pi}{8}$. [Art. 251.]

$-(\sqrt{2}+1) = -\cot\frac{\pi}{8} = -\tan\left(\frac{\pi}{2} - \frac{\pi}{8}\right)$.

From the first result, we get $\theta = n\pi + \frac{\pi}{8}$,

and from the second, $\theta = n\pi - \left(\frac{\pi}{2} - \frac{\pi}{8}\right) = n\pi - \frac{3\pi}{8}$,

both of which are included in $(8n-1)\frac{\pi}{8} \pm \frac{\pi}{4}$.

273. (1) $2\tan^{-1}\frac{1}{7} = \tan^{-1}\dfrac{\frac{2}{7}}{1-\frac{1}{7^2}} = \tan^{-1}\frac{14}{48} = \tan^{-1}\frac{7}{24} = \cos^{-1}\frac{24}{25}$,

$4\tan^{-1}\frac{1}{3} = 2\tan^{-1}\dfrac{\frac{2}{3}}{1-\frac{1}{9}} = 2\tan^{-1}\frac{3}{4} = \tan^{-1}\dfrac{\frac{3}{2}}{1-\frac{9}{16}} = \tan^{-1}\frac{24}{7}$

$= \sin^{-1}\frac{24}{25}$.

Thus each side of the identity $= \frac{24}{25}$.

(2) $\dfrac{3\sin 2\alpha}{5+3\cos 2\alpha} = \dfrac{6\tan\alpha}{1+\tan^2\alpha} \div \left\{5 + \dfrac{3(1-\tan^2\alpha)}{1+\tan^2\alpha}\right\} = \dfrac{3\tan\alpha}{4+\tan^2\alpha}$;

$\therefore$ first side of the identity $= \tan^{-1}\dfrac{3\tan\alpha}{4+\tan^2\alpha} + \tan^{-1}\left(\dfrac{\tan\alpha}{4}\right)$

$= \tan^{-1}\dfrac{\frac{3\tan\alpha}{4+\tan^2\alpha} + \frac{\tan\alpha}{4}}{1 - \frac{3\tan^2\alpha}{4(4+\tan^2\alpha)}} = \tan^{-1}\dfrac{16\tan\alpha + \tan^3\alpha}{16+\tan^2\alpha}$

$= \tan^{-1}(\tan\alpha) = \alpha.$

274. We have $$\frac{s(s-a)}{bc}=\frac{b^2+c^2}{4bc};$$

$$\therefore\ 2s(2s-2a)=b^2+c^2;$$

or $$(b+c+a)(b+c-a)=b^2+c^2,$$

that is, $$b^2+c^2+2bc-a^2=b^2+c^2;$$

$$\therefore\ \frac{a^2}{2}=bc,$$

which proves the proposition.

275. We have $$\sin B=\frac{b}{a}\sin A=\frac{119\sin 50^\circ}{97},$$

$$\begin{aligned}
\log 119&=2{\cdot}0755470\\
\log \sin 50^\circ&=\bar{1}{\cdot}8842540\\
&\overline{1{\cdot}9598010}\\
\log 97&=1{\cdot}9867717\\
\log \sin B&=\bar{1}{\cdot}9730293\\
\log \sin 70^\circ&=\bar{1}{\cdot}9729858\\
&\overline{\quad 435}
\end{aligned}$$

diff. for $1'=460$,

$$\frac{435}{460}\times 60''=57'';$$

$\therefore\ B=70^\circ\,0'\,57''$, or $109^\circ\,59'\,3''$, both values being admissible since $a<b$;

$$\therefore\ C=59^\circ\,59'\,3''\ \text{or}\ 20^\circ\,0'\,57''.$$

276. The $\angle BD_1F_1=\frac{1}{2}(\text{supp}^t\ \text{of}\ \angle F_1BD_1)=\frac{B}{2}=\angle BF_1D_1$.

Similarly $$\angle CD_1E_1=\frac{C}{2};$$

$$\therefore\ \angle F_1D_1E_1=180^\circ-\frac{B+C}{2}=90^\circ+\frac{A}{2}.$$

Again from the isosceles triangle AE_1F_1,

$$\angle AF_1E_1=90^\circ-\frac{A}{2};\ \therefore\ \angle D_1F_1E_1=90^\circ-\frac{A+B}{2}=\frac{C}{2},$$

and similarly $$\angle D_1E_1F_1=\frac{B}{2}.$$

Now $$r_a=\frac{E_1F_1\sin\frac{1}{2}E_1\sin\frac{1}{2}F_1}{\cos\frac{1}{2}D_1}=\frac{2s\sin\frac{A}{2}\sin\frac{B}{4}\sin\frac{C}{4}}{\cos\left(45^\circ+\frac{A}{4}\right)}$$

$$=4s\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{C}{4}\cdot\frac{\cos\frac{A}{4}}{\frac{1}{\sqrt{2}}\left(\cos\frac{A}{4}-\sin\frac{A}{4}\right)}$$

$$=4\sqrt{2}s\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{C}{4}\cdot\frac{1}{1-\tan\frac{A}{4}};$$

$$\therefore \frac{1}{r_a} : 1 - \tan\frac{A}{4} = 1 : 4\sqrt{2s}\sin\frac{A}{4}\sin\frac{B}{4}\sin\frac{C}{4}$$

$$= \frac{1}{r_b} : 1 - \tan\frac{B}{4} = \frac{1}{r_c} : 1 - \tan\frac{C}{4}$$

by symmetry.

277. The expression $= \tan^{-1}\dfrac{x\cos\theta}{1 - x\sin\theta} - \tan^{-1}\dfrac{x - \sin\theta}{\cos\theta}$

and this reduces to $\tan^{-1}\left\{\dfrac{\sin\theta\,(1 - 2x\sin\theta + x^2)}{\cos\theta\,(1 - 2x\sin\theta + x^2)}\right\}$,

which equals $\tan^{-1}(\tan\theta)$, or θ.

278. By Example XIII. *c*. 8 we have

$$\frac{\cos A + \cos B}{4\sin^2\dfrac{C}{2}} = \frac{a+b}{2c};$$

$\therefore\ a+b=2c$; whence a, c, b are in A.P.

279. The expression

$= 2\cos\alpha\cos\beta\,\{\cos(\gamma+\delta) + \cos(\gamma-\delta)\} + 2\sin\alpha\sin\beta\,\{\cos(\gamma-\delta) - \cos(\gamma+\delta)\}$

$= 2\cos(\gamma+\delta)\cos(\alpha+\beta) + 2\cos(\gamma-\delta)\cos(\alpha-\beta)$

$= \cos(\alpha+\beta+\gamma+\delta) + \cos(\alpha+\beta-\gamma-\delta) + \cos(\alpha-\beta+\gamma-\delta) + \cos(\alpha-\beta-\gamma+\delta).$

280. The $\angle BIC = 180° - \dfrac{B+C}{2} = 90° + \dfrac{A}{2}$;

$$\therefore\ \rho_1 = \frac{a}{2\sin BIC} = \frac{a}{2\cos\dfrac{A}{2}};$$

$$\therefore\ \rho_1\rho_2\rho_3 = \frac{abc}{8\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}} = \frac{R^3\sin A\sin B\sin C}{\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}}$$

$$= 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2rR^2,$$

since $$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

281. This is a particular case of Ex. 13 in XVII. *a*.

282. The equation may be written

$$b^2 \sin^2 2\theta = (c - a\cos 2\theta)^2,$$

or $$b^2(1-\cos^2 2\theta) = c^2 - 2ac\cos 2\theta + a^2\cos^2 2\theta,$$

that is, $$(a^2+b^2)\cos^2 2\theta - 2ac\cos 2\theta + c^2 - b^2 = 0;$$

therefore, by the theory of quadratic equations,

$$\cos 2\alpha + \cos 2\beta = \frac{2ac}{a^2+b^2};$$

$$\therefore\ 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 = \frac{2ac}{a^2+b^2};$$

whence $$\cos^2\alpha + \cos^2\beta = \frac{a^2+ac+b^2}{a^2+b^2}.$$

283. We have $c^2 = a^2+b^2-2ab\cos C$

$$= 2+2+\sqrt{2} - 2\sqrt{2}\,\sqrt{2+\sqrt{2}}\,.\,\frac{\sqrt{2+\sqrt{2}}}{2}$$

$$= 4+\sqrt{2} - \sqrt{2}\,(2+\sqrt{2}) = 2-\sqrt{2};$$

$$\therefore\ c = \sqrt{2-\sqrt{2}}.$$

Now $$\sin A = \frac{a}{c}\sin C = \frac{\sqrt{2}}{\sqrt{2-\sqrt{2}}},\ \frac{\sqrt{2-\sqrt{2}}}{2} = \frac{1}{\sqrt{2}};$$

therefore $A = 45°$, or $135°$, and since a is not the greatest side the smaller value must be taken.

Therefore $$B = 112\tfrac{1}{2}°.$$

284. We have $$\sin 3A = 3\sin A - 4\sin^3 A;$$

$$\therefore\ \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin^3 A;$$

$$\therefore\ \Sigma \sin^3 A = \frac{3}{4}\Sigma \sin A - \frac{1}{4}\Sigma \sin 3A.$$

Now $$\Sigma \sin A = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

and $$\Sigma \sin 3A = 2\sin\frac{3(A+B)}{2}\cos\frac{3(A-B)}{2} + 2\sin\frac{3C}{2}\cos\frac{3C}{2}$$

$$= 2\sin\left(270° - \frac{3C}{2}\right)\cos\frac{3(A-B)}{2} + 2\sin\frac{3C}{2}\cos\frac{3C}{2}$$

$$= -2\cos\frac{3C}{2}\cos\frac{3(A-B)}{2} + 2\sin\frac{3C}{2}\cos\frac{3C}{2}$$

$$= 2\cos\frac{3C}{3}\left\{\sin\frac{3C}{2} - \cos\frac{3(A-B)}{2}\right\}$$

$$= 2\cos\frac{3C}{2}\left\{-\cos\frac{3(A+B)}{2} - \cos\frac{3(A-B)}{2}\right\}$$

$$= -4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2};$$

$$\therefore\ \Sigma\sin^3 A = 3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - \cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

285. Take the third figure on p. 131, and first suppose that

$$\angle AB_2C = 2\angle AB_1C.$$

Then it easily follows that $\triangle CB_1B_2$ is equilateral;

$$\therefore\ b\sin A = a\sin B \text{ becomes } b\sin A = \frac{\sqrt{3}}{2}a.$$

Secondly, suppose that $\angle ACB_1 = 2\angle ACB_2$.

Then
$$\angle ACB_2 = \angle B_1 - \angle A = \angle B_2CB_1 = 180^\circ - 2\angle B_1;$$
$$\therefore\ 3B_1 = 180^\circ + A,$$
$$\sin 3B_1 + \sin A = 0;\ \text{or}\ 3\sin B_1 - 4\sin^3 B_1 + \sin A = 0.$$

Substituting $\frac{b}{a}\sin A$ for $\sin B_1$, and reducing we obtain the required result.

286. If we write $\sqrt{y}$ in the place of x the resulting equation has $\tan^2\alpha$, $\tan^2\beta$, $\tan^2\gamma$ for its roots. If in the last equation we further write z for $1+y$ the resulting equation has $\sec^2\alpha$, $\sec^2\beta$, $\sec^2\gamma$ for its roots.

After making the above substitutions the equation in z is

$$z^3 - (p^2+3)z^2 + (4p^2 - 2pr + 3)z - (p-r)^2 - 1 = 0;$$
$$\therefore\ \sec^2\alpha\sec^2\beta\sec^2\gamma = \text{product of the roots}$$
$$= (p-r)^2 + 1.$$

Otherwise. Let t_1, t_2, t_3 be the roots of the given equation; then

$$\sec^2\alpha\sec^2\beta\sec^2\gamma = (1+t_1^2)(1+t_2^2)(1+t_3^2)$$
$$= 1 + \Sigma t_1^2 + \Sigma t_1^2t_2^2 + t_1^2t_2^2t_3^2,$$

where
$$\Sigma t_1 = p,\ \Sigma t_1t_2 = 0,\ t_1t_2t_3 = r.$$

Thus
$$\sec^2\alpha\sec^2\beta\sec^2\gamma = 1 + p^2 - 2pr + r^2.$$

287. Square the given equation, and write it in the form

$$\sin^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\sin^2\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\sin^2\left(\frac{\pi}{4} - \frac{\gamma}{2}\right)$$
$$= \cos^2\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\cos^2\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\cos^2\left(\frac{\pi}{4} - \frac{\gamma}{2}\right),$$

or $(1-\sin\alpha)(1-\sin\beta)(1-\sin\gamma)=(1+\sin\alpha)(1+\sin\beta)(1+\sin\gamma)$;

$$\therefore \sin\alpha+\sin\beta+\sin\gamma+\sin\alpha\sin\beta\sin\gamma=0,$$

or $$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+8\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}=0;$$

hence $$1+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=0,$$

that is, $$1+\frac{1}{2}(\cos\alpha+\cos\beta+\cos\gamma-1)=0;$$

$$\therefore \cos\alpha+\cos\beta+\cos\gamma+1=0.$$

288. Each side of the heptagon subtends an angle $\frac{\pi}{7}$ at the circumference of the circle whose diameter is 2. Therefore if x represent a side,

$$x=2\sin\frac{\pi}{7}.$$

Now by Art. 331, the roots of the equation

$$8y^3+4y^2-4y-1=0,$$

are $$\cos\frac{2\pi}{7},\quad \cos\frac{4\pi}{7},\quad \cos\frac{6\pi}{7}.$$

Therefore $2\cos\frac{2\pi}{7}$ satisfies $y^3+y^2-2y-1=0$.

Put $y=2-4\sin^2\frac{\pi}{7}=2-x^2$ in this equation.

We obtain, after reduction,

$$x^6-7x^4+14x^2-7=0,$$

the roots of which are $2\sin\frac{\pi}{7},\quad 2\sin\frac{2\pi}{7},\quad 2\sin\frac{3\pi}{7}$.

The first of these values corresponds to a side of the heptagon, the second and third to chords subtending at the circumference angles of $\frac{2\pi}{7}$ and $\frac{3\pi}{7}$ respectively. That is they represent the diagonals of the heptagon, as is easily seen from a figure.

289. We have $$\cot(A+C)=-\cot B=-1;$$

$$\therefore \frac{\cot A\cot C-1}{\cot A+\cot C}=-1;$$

that is, $$1+\cot A+\cot C+\cot A\cot C=2,$$

or $$(1+\cot A)(1+\cot B)=2.$$

290. We have $8R^2 = a^2 + b^2 + c^2$

$$= (2R\sin A)^2 + (2R\sin B)^2 + (2R\sin C)^2;$$

$$\therefore \sin^2 A + \sin^2 B + \sin^2 C = 2;$$

$$\therefore 1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C = 4,$$

$$\cos 2A + \cos 2B + \cos 2C + 1 = 0;$$

$$\therefore 4\cos A \cos B \cos C = 0; \qquad \text{[Examples XII. } d. \text{ 9.]}$$

therefore one of the angles must be a right angle.

291. We have $\sin B = \cos A \cos B$

$$= \cos A \tan C,$$

and $\cos B = \tan C;$

$$\therefore 1 = \tan^2 C\,(1 + \cos^2 A);$$

$$\cos^2 C = (1 - \cos^2 C)\,(1 + \cos^2 A),$$

$$\frac{\sin^2 A}{\cos^2 A} = \left(1 - \frac{\sin^2 A}{\cos^2 A}\right)(2 - \sin^2 A);$$

or $\sin^2 A = (1 - 2\sin^2 A)\,(2 - \sin^2 A);$

whence $\sin^4 A - 3\sin^2 A + 1 = 0;$

$$\therefore \sin^2 A = \frac{3 - \sqrt{5}}{2},$$

the other value being impossible;

$$\therefore \sin^2 A = \frac{1}{4}(6 - 2\sqrt{5}),$$

or $$\sin A = \frac{\sqrt{5} - 1}{2} = 2\sin 18^\circ$$

$$= \sin B = \sin C \text{ similarly.}$$

292. It will be sufficient to prove that

$$ab + bc + ca = r^2 + s^2 + 4Rr \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$abc = 4Rrs \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now $r^2 + s^2 + 4Rr = r(r_1 + r_2 + r_3) + s^2$ [XVIII. *a*. Ex. 25]

$$= \left(\frac{\Delta^2}{s(s-a)} + \ldots + \ldots\right) + s^2$$

$$= \{(s-b)(s-c) + \ldots + \ldots\} + s^2$$

$$= 4s^2 - 2s(a+b+c) + bc + ca + ab$$

$$= bc + ca + ab.$$

Again, $4Rrs = \frac{abc}{\Delta} \cdot \Delta = abc.$

293. We have $\sin\frac{\pi}{14}=\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)=\cos\frac{3\pi}{7}$,

which is a root of the equation

$$8x^3-4x^2-4x+1=0.$$

See solution of XXV. *c*. Ex. 16.

294. The distances of the successive heaps from the starting point are

$$2r\sin\frac{\pi}{n},\quad 2r\sin\frac{2\pi}{n},\quad 2r\sin\frac{3\pi}{n},\ \ldots\ldots\ 2r\sin\frac{(n-1)\pi}{n};$$

$\therefore$ the whole distance traversed is twice the sum of this series.

Now the sum of the sines $=\dfrac{\sin\frac{(n-1)\pi}{2n}\sin\frac{1}{2}\left\{\frac{\pi}{n}+\frac{(n-1)\pi}{n}\right\}}{\sin\frac{\pi}{2n}}$ [Art. 296.]

$$=\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\sin\frac{\pi}{2}}{\sin\frac{\pi}{2n}}=\cot\frac{\pi}{2n},$$

whence the required result follows.

295. $\cos\frac{\pi}{15}\cos\frac{4\pi}{15}=\frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{\pi}{5}\right)=\frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right)=\frac{3+\sqrt{5}}{8}$;

$\cos\frac{2\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{2}\left(\cos\frac{\pi}{3}+\cos\frac{3\pi}{5}\right)=\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right)=\frac{3-\sqrt{5}}{8}$;

$\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}=\frac{1}{2}\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right)=\frac{1}{2}\left(\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right)=\frac{1}{4}$;

$\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}$;

$\therefore$ multiplying these results together, we have

$$\frac{1}{8}\left(\frac{3+\sqrt{5}}{8}\right)\left(\frac{3-\sqrt{5}}{8}\right)=\frac{4}{8^3}=\frac{2^2}{2^9}=\left(\frac{1}{2}\right)^7.$$

296. We have $ax+by+cz=2\Delta$.

Now $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2$

$$=(bx-ay)^2+(cy-bz)^2+(az-cx)^2;$$

or $(a^2+b^2+c^2)(x^2+y^2+z^2)-4\Delta^2=(bx-ay)^2+(cy-bz)^2+(az-cx)^2$.

$\therefore$ $x^2+y^2+z^2$ is a minimum when the expression on the right is zero; that is when

$$bx=ay,\ cy=bz,\ az=cx;$$

$$\therefore\ \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{ax+by+cz}{a^2+b^2+c^2}=\frac{2\Delta}{a^2+b^2+c^2}.$$

297. With the notation and fig. of Art. 231, suppose AD and BC intersect in E, then r_a is the radius of the escribed circle opposite to E in the ΔABE;

$$\therefore r_a = \frac{a\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{A+B}{2}},$$

$$\therefore \frac{a}{r_a} = \tan\frac{A}{2} + \tan\frac{B}{2};$$

$$\therefore \frac{a}{r_a} + \frac{c}{r_c} = \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \tan\frac{D}{2}$$

$$= \frac{b}{r_b} + \frac{d}{r_d}, \text{ similarly.}$$

298. Draw AH, AH' perpendicular to BC; then

$$\angle PAH = (90° - B) - (90° - C) = C - B,$$

and

$$AH = 2R\sin B\sin C,$$

$$\therefore AP = \frac{2R\sin B\sin C}{\cos(C-B)};$$

$$\therefore \frac{1}{AP} + \ldots + \ldots = \frac{1}{2R}\left\{\frac{\cos(B-C)}{\sin B\sin C} + \ldots + \ldots\right\}$$

$$= \frac{1}{4R}\cdot\frac{(\sin 2C + \sin 2B) + \ldots + \ldots}{\sin A\sin B\sin C}$$

$$= \frac{2}{R}. \qquad \text{[Art. 135, Ex. 1.]}$$

Again, $BA' = 2R\cos BA'A = 2R\cos C$, since B, A', A, C are concyclic,

$$\therefore A'H' = 2R\cos B\cos C;$$

$$\therefore \frac{1}{A'P} + \ldots + \ldots = \frac{1}{2R}\left\{\frac{\cos(C-A)}{\cos B\cos C} + \ldots + \ldots\right\}$$

$$= -\frac{1}{2R}\frac{(\cos 2C + \cos 2B) + \ldots + \ldots}{\cos A\cos B\cos C}$$

$$= -\frac{2(\cos 2A + \cos 2B + \cos 2C)}{4R\cos A\cos B\cos C}$$

$$= \frac{1}{2R}\cdot\frac{4\cos A\cos B\cos C + 1}{\cos A\cos B\cos C}. \qquad \text{[XII. } d\text{. Ex. 9.]}$$

299. Let $\tan\frac{\theta}{2}=t$, then $\sin\theta=\frac{2t}{1+t^2}$, $\cos\theta=\frac{1-t^2}{1+t^2}$; substitute these values in the given equation; then after reduction, we obtain

$$t^4b+2t^3(c-a)-2t(c+a)-b=0.$$

This equation has *four* roots; three of which are

$$\tan\frac{\alpha}{2},\ \tan\frac{\beta}{2},\ \tan\frac{\gamma}{2};$$

also $$t_1t_2t_3t_4=-1,\quad t_2t_3+t_3t_4+\ldots=0.$$

Eliminating t_4 by means of these equations,

$$t_2t_3+t_3t_1+t_1t_2=\frac{1}{t_2t_3}+\frac{1}{t_3t_1}+\frac{1}{t_1t_2};$$

$$\therefore\ \tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$$

$$=\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\gamma}{2}\cot\frac{\alpha}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2};$$

that is, $$\Sigma\left(\tan\frac{\beta}{2}\tan\frac{\gamma}{2}-\cot\frac{\beta}{2}\cot\frac{\gamma}{2}\right)=0;$$

$$\therefore\ \Sigma\frac{\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2}-\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=0;$$

$$\therefore\ \Sigma\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=0;$$

$$\therefore\ \Sigma(\cos\beta+\cos\gamma)\sin\alpha=0;$$

or $$\Sigma\sin(\alpha+\beta)=0.$$

300. By Example 2, Art. 331,

$$\sec^2\frac{\pi}{7},\quad \sec^2\frac{2\pi}{7},\quad \sec^2\frac{3\pi}{7}$$

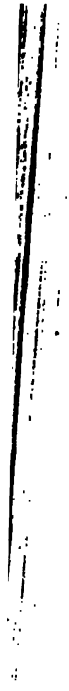
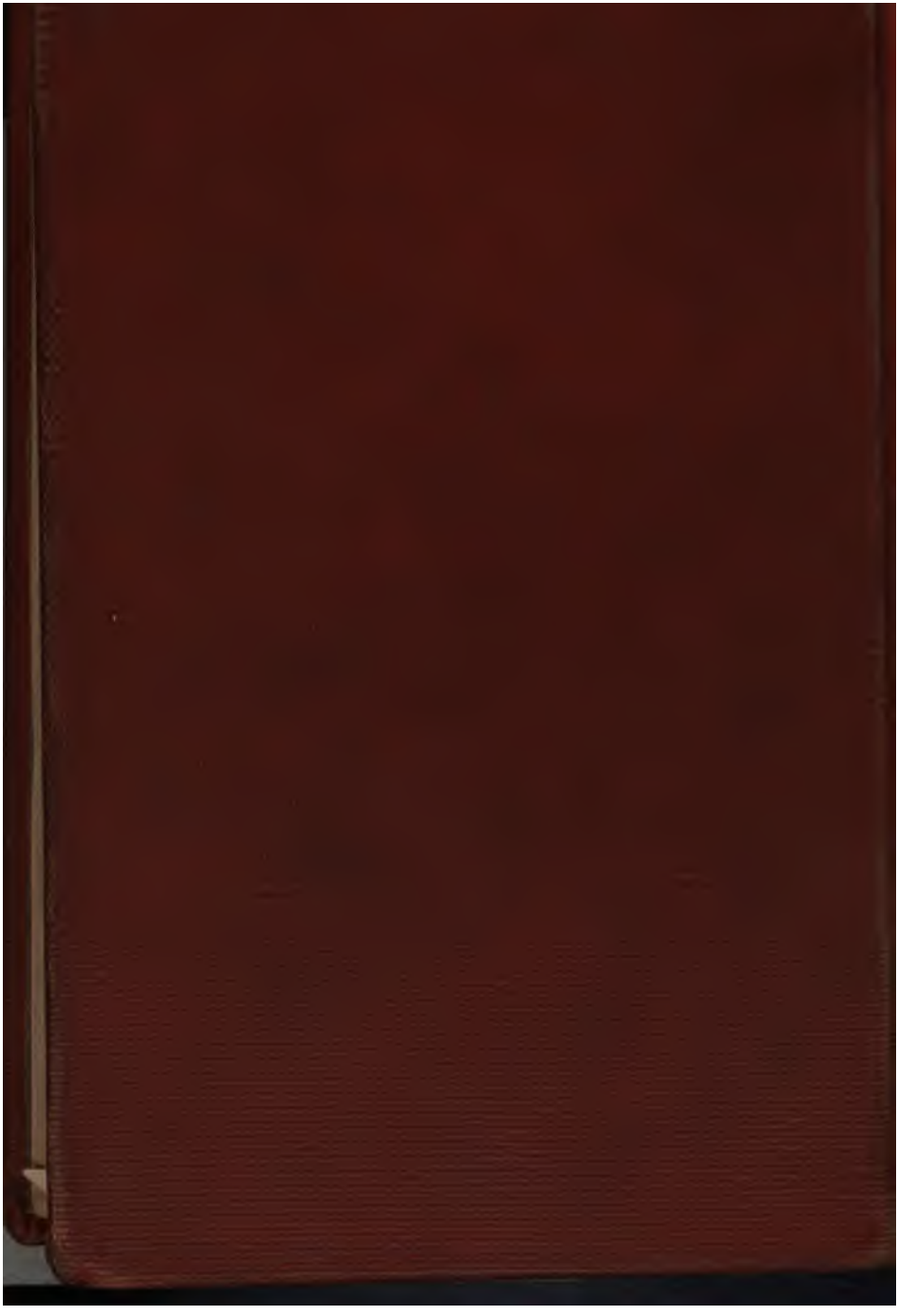
are roots of the equation

$$x^3-24x^2+80x-64=0;$$

$$\therefore\ \sec^2\frac{\pi}{7}+\sec^2\frac{2\pi}{7}+\sec^2\frac{3\pi}{8}=24.$$

Also from XXV. *c*. Ex. 21,

$$\operatorname{cosec}^2\frac{\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}=8,$$

whence the required result follows.

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